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ABSTRACT

Eleven research reports in the area of models of learning mathematics are presented in this publication of the Mathematics Education Reports series. The papers represent a mixture of theories, viewpoints, and references to other areas. Content areas addressed range from preschool to college levels. All the papers are concerned with the learning of mathematics and all are attempts to clarify and apply theory through the mechanism of models. The papers rely on observations of students and have possible implications for classroom instruction. Topics include: (1) modeling and theorizing in mathematics education; (2) experimental models for the child's acquisition of counting and of addition and subtraction; (3) a logical-cognitive view of learning and teaching mathematical concepts; (4) instructional and theoretical implications of a mathematical model of cognitive development; (5) a learning model for some algebraic concepts; (6) abstraction, generalization, and the design of mathematical experiences for children; (7) towards a model for the teaching of mathematics as goal-directed activity; (8) preliminary steps toward a cognitive model of learning primary mathematics; (9) intraconcept analysis of rational number addition and subtraction, indirect validation studies; (10) an organizational model of abilities related to inference in mathematical contexts; and (11) logical reasoning learning hierarchies. (MP)

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EXPLORATIONS IN THE MODELING OF THE LEARNING OF MATHEMATICS

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MATHEMATICS EDUCATION REPORTS

The Mathematics Education Reports series makes available recent analyses and syntheses of research and development efforts in mathematics education. As a part of this series, we are pleased to make available this collection of research reports in the area of models of learning mathematics. The eleven papers presented here were coordinated by the Models Working Group of the Georgia Center for the Study of Learning and Teaching Mathematics.

Other Mathematics Education Reports make available information concerning mathematics education documents analyzed at the ERIC Information Analysis Center for Science, Mathematics, and Environmental Education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Special bibliographies announce the availability of documents and review the literature in selected interest areas of mathematics education.

Priorities for the development of future Mathematics Education Reports are established by the advisory board of the Center, in cooperation with the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, and other professional groups in mathematics education. Individual comments on past Reports and suggestions for future Reports are always welcome.

Jon L. Higgins
Associate Director for
Mathematics Education

Acknowledgements

The editors would like to thank Suzanne Damarin and Lars Jansson for their very considerable efforts in acting as in-house referees on the papers presented in this volume. Likewise, we wish to thank Leslie P. Steffe, University of Georgia, for providing encouragement and assistance through the Georgia Center for the Study of Learning and Teaching Mathematics. All papers in this volume were submitted to anonymous referees selected by the Georgia Center. We particularly appreciate their services and regret that we cannot thank them by name.

The members of the Models for Learning Mathematics Working Group have contributed greatly toward improving these papers through their discussions and presentations at group meetings. The authors are to be commended for perseverence in improving the quality of their papers through reactions and rewrites in response to group discussions and to sometimes copious editor and referee suggestions and questions.

Finally, a special thank you goes to Nan Collins and Alice Greenleaf for typing and retyping the many versions of this monograph.

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Foreword

The Georgia Center for the Study of Learning and Teaching Mathematics is a consortium of investigators dedicated to the improvement of the mathematical education of children. The consortium is currently organized into three Projects and ten Working Groups, listed below. Each Project and Working Group has general and specific goals. The general goals are common and constitute the goals of the Center as an organization. These goals were established as a result of a meeting of the Working Group Leaders in October of 1976. Specific goals are emerging for each Working Group. The goals of the Center are listed below.

(1) Establish and maintain a consortium of investigators dedicated to the improvement of mathematical education in the elementary and secondary schools through disciplined inquiry.

(2) Identify fundamental problems and issues within mathematics education.

Projects and Leaders	Conceptual Development of Mathematics	Problem Solving	Teaching Strategies
	Leslie P. Steffe	Larry Hatfield	Thomas J. Cooney
Working Groups and Leaders	Applications	Combinations of Heuristics	Analyzing Teacher Behavior
	Richard Lesh	J. Philip Smith	Thomas J. Cooney
	Models for Learning Mathematics	Task Variables	Protocols Development
	William Geeslin	Gerald Kulm	Kenneth Retzer
	Number and Measurement		Teaching Strategies
	Grayson Wheatley		John Dossey
	Rational Numbers		
	Thomas Kieren		
	Space and Geometry		
	Izzie Weinzweig		

(3) Conduct studies in mathematics education which contribute significantly to the organization of a particular field of investigation or to the resolution of fundamental problems and issues.

(4) Promote interinstitutional cooperation which most efficiently utilizes available human resources to address fundamental problems and issues in mathematics education.

(5) Promote communication among investigators in mathematics education and allied disciplines through publication for Center studies.

(6) Translate knowledge in mathematics education into a form useable by school practitioners.

This monograph represents a significant step toward realization of Goals 1-5 above. The investigators of the Models for Learning Mathematics Working Group are to be congratulated for their intensive efforts during the time the studies were in progress. A great deal of effort also has been expended by the faculty and students at the University of New Hampshire in preparation of the monograph. It is only through such dedicated efforts that coordinated research can be conducted. The investigators of the Center view the utilization by school practitioners of knowledge generated through research efforts as both desirable and realistic. The next immediate step to be taken after the publication of a monograph such as this one is to generate interpretative reports of the theory and research. The Center Policy Board urges any individual interested in such interpretative reports to collaborate with the Center in preparation of the documents.

The Policy Board of the Center would like to thank ERIC/SMEAC of The Ohio State University for publishing this and various other monographs of the Center. This cooperation has greatly facilitated ongoing studies and has made the monographs published available to any interested individual. Other monographs and working papers are available and may be obtained by writing any member of the Policy Board.

The Policy Board

Leslie P. Steffe, Director

Thomas J. Cooney

Larry L. Hatfield

Jeremy Kilpatrick

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Modeling is an activity, a purposive behavior, whose significance is determined by the theoretical basis of the research framework. Building a model, or employing an already available model, must occur within the structure of a theory. Through a survey of the development of models in the nineteenth century this paper distinguishes several essential features of models. Models establish a partial analogy which depends equally on a clear similarity, and an obvious difference, between the model and what is being modeled. A model typically is a temporary explanatory device which allows the researcher to simplify, visualize, and idealize what is being modeled. This paper places the use of models in mathematics education within a broader research framework which provides for deeper understanding of the methodological benefits and limitations of modeling.

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The purpose of this paper is to present a logical-cognitive point of view concerning the relationships among mathematical concepts, the psychology of the learner, and instruction. The developmental psychology of Piaget is the basic background from which this model emerges. Two classifications of mathematical concepts are identified: figural concepts and operational concepts. Various combinations of these concepts are hypothesized to form internalized conceptual structures. The study of the ways in which these structures are formed and function is assumed to be the very essence of the psychology of learning mathematical concepts--to comprehend or understand a concept means to assimilate it into an appropriate structure. An analogy is drawn between mathematical (conceptual) structures and Piaget's operational structures, and their corresponding acquisitions. In particular, simple abstraction is applied to figural concept acquisition and reflective abstraction is applied to operational concept acquisition. Piaget's periods of mental development are outlined and used as motivation in proposing three learning-instructional phases of concept acquisition. These phases are referred to as exploration, assimilation, and formalization.

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When taught formally, the acquisition of these concepts represents for the student a problem of accommodation. Didactic reversal attempts to achieve this accommodation through an assimilative process by building meaning for the new algebraic forms on the basis of the pupil's existing cognition. For the concept of equations in one unknown, this is achieved by introducing arithmetic identities and transforming them into equations. For the concept of linear equations in two variables, this implies formalizing the student's concept of the straight line in the Cartesian plane. It is only when the algebraic forms have acquired meaning that reversal is encouraged. That is, starting now from meaningful algebraic forms, the student is asked to find their arithmetic or geometric representations.

These learning schemes are based on the assumption that there are different modes of understanding mathematics and that their integration is essential to any pedagogical presentation. It is for this purpose that the article reviews three models of understanding which describe types of understanding of mathematics.

Abstraction, Generalization, and the Design of Mathematical Experiences for Children, *Thomas R. Post and Robert E. Reys*........ 117

This paper suggests a model for conceptualizing mathematical concepts. It emphasizes two important aspects of conceptual development: abstraction and generalization.

The variability principles suggested by Dienes form the conceptual framework from which the ideas in this paper emanate. Perceptual variability was hypothesized by Dienes to promote abstraction of a mathematical concept and mathematical variability was similarly hypothesized to promote generalization of that same concept. The ideas developed are an attempt to provide a conceptual framework within which these psychological principles can be utilized in the design and implementation of instructional activities within the classroom setting. As will be seen, the individual exemplars are topic or concept specific, although the procedures for their development remain consistent throughout. These developmental procedures can be applied to a wide variety of mathematical concepts.

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An explanation and definition of ordering theory, a deterministic measurement model, are presented. This theory is then applied in a replication study and an extension study in order to construct logical reasoning learning hierarchies based upon Piaget's sixteen binary combinations. The first study, of 50 grade nine subjects, attempted to replicate an earlier study of Airasian, Bart, and Greaney. The second study extended the same investigation to subjects in grades seven and eleven. Logical reasoning hierarchies are presented for all three grade levels and comparisons are made with the replicated study.

SEARCHING FOR MODELS

William E. Geeslin
University of New Hampshire

The search for models of mathematics learning has been the result, at least partially, of attempts to create a "science" of mathematics education. The cogent assumption is that model construction and validation will lead to better understanding of children's mathematical learning which, in turn, could lead to more effective classroom instruction. To engage in the endeavor of model building, one necessarily assumes that the results of learning and teaching are predictable, at the very least, in a probabilistic manner. In simplistic terms one must choose between viewing classroom instruction (and its results) as an applied science or viewing instruction purely as an art. Use of models would seem to imply that a significant portion of classroom activity (school learning, if you prefer) is amenable to a scientific-like description and explanation. That is, this activity can be explained in a rational manner. The rational explanation should increase the predictability of the results of classroom instruction. At present it seems too strong an assumption to believe that deterministic models can be used for such a complex activity as human behavior. However, science-like assumptions are not totally unwarranted. Some progress has been made over the past decade in research in mathematics and science education. One manifestation of interest in research is the current emphasis in the National Science Foundation on research in science education.

But as Edgerton (Note 1) has stated, merely copying methods from the physical sciences can be an intellectual pitfall. Still, the use of models shows some promise of furthering the aims of mathematics education. Useful methods, procedures, and knowledge from a variety of disciplines should be relied upon when appropriate. It is clear that in its present form no single discipline offers complete answers concerning questions about classroom instruction, but several disciplines do offer viewpoints and methodological tools that may be adaptable to mathematics education research. The Models of Learning Mathematics Research Working Group has involved psychologists, mathematicians, and philosophers in model-building efforts. A single person cannot expect to master all of these disciplines. But mathematics educators can become knowledgeable in specific aspects of a variety of disciplines. Likewise, formation of inter-disciplinary teams of researchers should be encouraged. Theories and models of learning mathematics, if they are to exist, will have to be created through such interdisciplinary teams. The primary importance of the mathematics educators on such teams should not be underestimated. No other professional group possesses both the subject-matter knowledge and the interest in the content-psychological interaction necessary for focusing on mathematics classroom instruction as an entity in and of itself. Educational research that ignores the subject-matter or uses mathematics as a mere convenience, due to its structure, has not provided the desired results for mathematics education. Mathematics possesses a peculiar property of being an entirely human-constructed entity which also may be applied to a large variety of human-observed physical experiences. The mixture of internal consistency not requiring experimental validation and the applications of mathematics which do require outside validation prevent mathematical knowledge and its attainment from being amenable to any single discipline known today. Mathematics learning does provide mathematics educators with a vast field of observation from which to attempt the creation of a science.

Research in mathematics education has tended to be piecemeal, sporadic, and faddish. One often finds research that is neither long-term systematic research nor even a set of coordinated studies. Begle (1979) is one of the unusual examples of even reviewing mathematics education research in its own right. Mathematics education research has produced little in the way of theory. The papers in this volume represent an attempt to clarify theory and formulate models of mathematical learning. These papers also represent an attempt to clarify the very meaning of the terms "theory" and "model." Emphasis on statistical manipulation of empirical data frequently prevents pulbication of conceptual writing such as that found in this volume. If knowledge about learning and teaching is to be produced, both theoretical frameworks and hypothesized models of learning and teaching must be created. Replication, elaboration, refinement, and validation of educational applications using these models is necessary, of course, but one cannot study a nonexistent entity. In fact, lack of models prevents sound interpretation of the results of much present research.

The papers in this volume focus on exposition of particular models. Models can be used either to predict the outcomes of certain classroom activities or to simulate the actual process of cognition. As noted by Osborne (1976) and reinforced by Richards (1979), model construction is something more than connecting lines and boxes within a diagram. In fact, the reader is invited to test Richard's (1979) characterization of a model against the presentations in this monograph. The "models" presented here may fail this test on one or more characteristics. Nonetheless, the papers as a whole represent considerable progress toward formulation of models of mathematics learning. This type of exposition is a necessary first step in research. Note that each paper makes some use of observations of children. Model construction without consideration of the student is a pointless task. Likewise, gathering data on large numbers of students in the absence of any theoretical framework can be pointless. Model building precedes controlled experimental studies and both are important for progress in mathematics education research.

The reader of this volume should not expect any overarching theory or model. The models offered are incomplete and in some cases contradictory. As noted in the paper by Steffe, von Glasersfeld, and Richards, the models are usually content specific. While not always explicitly stated, each model depends on the background theory and the author's interpretation of that theory from which the model was postulated. This theory "colors" the model and actually prevents certain types of models from being employed. Steffe et al., Mick and Brazier, and Mierkiewicz all rely heavily on Piagetian theory. Post and Reys, Herscovics, and Fuson rely on a modified version of Piagetian or developmental theory. In contrast with this, Greeno's paper is derived within an information processing framework. The Uprichard and Phillips and Jansson papers rely almost solely on the work of Gagne. The Damarin paper is interesting in that it appears to address the teaching of logic from a developmental viewpoint. Examination of most papers reveals a mix of theories, viewpoints, and references to other areas such as linguistics or mathematics from a mathematician's viewpoint. Content areas addressed range from preschool to college levels. Yet all the papers are concerned with the learning of mathematics and all the papers are attempts to clarify and apply theory through the mechanism of models. The papers rely on observations of students and have possible implications for classroom instruction. In some cases such as Steffe et al., these implications depart radically from common practice. These papers are not mere arguments over statistical methodology. Richards (1979) notes that theories determine the data. They also determine the method of obtaining that data. Thus, the theory and model are the primary objects of our interest.

All the models in this volume are under development. Some models change on a monthly basis. The models are only partially documented at best and do not necessarily generalize beyond the classroom style in which they were developed. Explicitly stating the models was a worthwhile activity for the investigators. The review process used in compiling this volume has forced each researcher to note assumptions, clarify arguments, and explain theory. This process helps to sharpen and focus research. The fact that the models are evolving should not deter the reader from pursuing investigations or attempting applications of the ideas presented. The importance of this volume lies in the methods and style of explanation used by each author. Construction of models is long, hard work. This volume represents a beginning of this work and a methodology for proceeding. Even if a "correct" model is contained in this volume, validation and refinement are necessary and require more of the type of effort reflected in these papers. If, as noted earlier, a "science" of mathematics education is developed, it cannot be accepted without empirical investigation. It must be open to challenge and undergo continual development. In this sense these papers are a challenge to the mathematics education community. Disproof, alternate explanation, and change are not only welcome but they are sought fervently. Models do not have some mystic power that will lead us from the wilderness of ignorance. Nor can we expect an "Einstein" to reveal knowledge instantaneously. Employment of models and the associated methodology may assist in the understanding of mathematics learning if long-term, systematic observation and development are undertaken.

Reference Notes

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> Mathematical social science is, first and foremost, social science. If it is bad social science, the fact that it is good mathematics should provide little comfort.
>
> H.A. Simon, 1954, p. 388

MODELING AND THEORIZING IN MATHEMATICS EDUCATION

John Richards
University of Georgia

Two characteristics are common to all models:

(1) There is a similarity in structure between the model and what is being modeled (I call what is being modeled the "prototype," following Apostel, 1961). And

(2) there is a clear and obvious difference between the model and the prototype.

This provides the basis for, what I take to be, the fundamental benefit of working with models: A model allows the researcher to focus on specific structural attributes of the prototype. These attributes are reflected or represented in the model. And a clear separation of the model from the prototype allows the investigator to isolate the functions and effects of these attributes.

Some terminology adopted from Mary Hesse allows us to focus on these two fundamental features of a model. A model provides an analogy between the objects (primitives) of the prototype, and the corresponding primitives of the model. The extent to which they are the same is the positive analogy. The extent to which they differ is the negative analogy. Properties of the model which are undetermined comprise the neutral analogy (cf. Hesse, 1966, p. 8).

In addition to these two primary characteristics of all models there are secondary features which are common to most models:

(3a) A model often simplifies the situation. By replicating the structure alone, the model-maker has made a theoretical decision to ignore certain aspects of the situation being modeled.

(3b) A model is usually visualizable. It provides a picture or a realization.

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I am indebted to Suzanne Damarin, Karen Fuson, John Messer, Alan Osborne, Les Steffe, Gabriel Stolzenberg, and Ernst von Glasersfeld for their suggestions on earlier versions of this paper.

(3c) A model is often an ideal. It is used in the sense of a paradigm. It is something to be imitated, copied, or patterned.

(3d) A model is transitory. It is not the "real" thing, but a temporary device to be used and then cast aside. It is a "disappearing schema."

In this paper I justify the characterization of models in terms of these primary and secondary features, and place the use and function of models within a broader research framework. In so doing I offer an abridged survey of the literature. In the first part of the paper it is argued, philosophically, that these features are in fact common to the diverse uses of models. And, through a brief historical survey, it is argued that these qualities are just those which recommended the use of models in the first place. In contrast with other accounts, this survey emphasizes the similarities, rather than the differences, in the disparate use of models. Although not definitive, this does provide a basis for the above characterization and, furthermore, provides a deeper insight into the function of models.

In the second part of the paper I consider models from the perspective of the philosophy of science, and describe a framework for research which takes into account the perspective and role of the researcher. This alters our understanding of the nature of models by allowing us to focus on the activity of modeling. From this perspective, a model is seen as essentially directed towards the researcher. Modeling is purposive behavior. The significance of this activity, the purpose, is determined by the theoretical basis of the research framework. Building a model, or employing an already available model, must occur within the structure of a theory. Research itself is only meaningful to the extent that it is embedded in theory.

Finally, I examine the use of models in mathematics education as a part of the context of this broader research framework. In the literature in mathematics education, models are usually not distinguished from theories. This conflation is destructive in two ways. It diminishes the specific benefits gained by working with models, and, by obscuring the nature of the research framework, there is a fundamental misunderstanding of the role of the researcher.

The cross-disciplinary fabric of mathematics education research makes it particularly difficult to isolate the operative theory and to properly identify the methodological framework. Nevertheless, fruitful research in mathematics education requires the careful enunciation of background theories. This does not mean that researchers must shift their concern to the esoteric development of novel theories. Rather, it emphasizes the responsibility of the researcher to clarify the methodological and theoretical assumptions already implicitly operating in the research. The construction of viable models in mathematics education depends on clarifying the prototype, and thereby establishing the extent of the positive analogy and the limitations inherent in the negative analogy. This clarification can only be done within a clear theoretical framework.

The Nature of Models

In this part I justify and explicate the primary and secondary characteristics of models. I then consider, in turn, some of the developments leading to their use in physics, mathematics and logic, and, more recently, in applied mathematics, the social sciences, and cybernetics. Whereas other treatments of the nature of models have emphasized the differences among the diverse uses of models, this study emphasizes what is held in common.

Primary and Secondary Characteristics of Models

The oldest use of model is captured by the notion of a scale model, or model boat (obviously these are older than model cars and airplanes). The word model derives from the Latin modulus meaning small measure. Thus the concept appears to originate as a miniature representation of something. It is a representation insofar as it preserves structure, but it is a miniature. The scale model is similar in structure (or isomorphic) to the prototype, but there is an obvious difference in scale and, generally, in materials used for the construction. A model boat has the same proportions, or internal relationships, or structure, but it has different external relationships. The obviousness of the negative analogy is marked in the name scale model itself.

From this initial usage, the concept is extended in several directions. Because it preserves the structure of the prototype, a model is often taken as a pattern for something to be made. And similarly, it is taken as an example for imitation, or emulation. But even when a model serves as a paradigm, it is essential to emphasize the negative analogy. In learning foreign verb endings we are given a model. This is useful only insofar as we distinguish those aspects of the model which are to be emulated. Clearly, only the endings are to be copied. In this way the particular verb chosen can serve as a model, even though it is an instance -- a prototype. When the verb functions as a model, it is essential to separate out those features which are not to be emulated. The negative analogy must be apparent for the verb to serve as a model.

A model differs from a replica which is intended to be an exact reproduction, and may even be confused with the original. It also differs from a facsimile where the same materials are used in the original and the model. A model is intended to be obviously different from the object being modeled. "A model is not a picture of a system, but a device for the attainment or formulation of knowledge about it" (Hesse, 1966, p. 286). There is a necessary difference in all models and their prototypes. This is not only acceptable, it is expected.

These primary characteristics of a model -- the only partial similarity with the prototype -- is precisely what recommends models. As argued by Rosenblueth and Weiner (1945):

> No substantial part of the universe is so simple that it can be grasped and controlled without abstraction. Abstraction consists in replacing the part of the universe under consid-

> eration by a model of similar but simpler structure. Models ... are thus a central necessity of scientific procedure (p. 316, emphasis added).

The model is similar, but it is different, simpler. The differences, the negative analogy, must be clearly set forth and the model must be carefully separated from the prototype.

Failure to heed this separation is the original sin of model construction. As pointed out by Ernest Nagel:

> ... a model may be a potential intellectual trap as well as an invaluable intellectual tool. The chief dangers are twofold: some inessential features of a model may be mistakenly assumed to constitute an indispensable feature of the theory embedded in it; and the model may be confused with the theory itself (Nagel, 1961, p. 115).

The secondary, or derivative, characteristics of a model function mainly to emphasize the primary characteristics. These are common to most models. First, (3a) a model often simplifies the situation. This brings out a structure for the prototype, while ignoring what, in this context, are seen as idiosyncrasies of the prototype. This is a theoretical decision. The particular theory determines what is structure and what is chaff. When a model is used to extend a theory to a complex practical situation, it simplifies the real situation according to principles determined by the theory. The model then contains only a limited number of carefully chosen features of the situation, and operates by introducing a ceteris paribus clause (all other things being equal).

Second, (3b) a model is often used to help visualize something that is unobservable because, for example, it is too large, or too small, or too distant, or too complex, or not constructed yet. A fashion model is used to enable someone to visualize how the clothes look on a person rather than on a rack. A scale model allows someone to see a large boat from a different perspective, as a total unit. And a dynamical model of a mathematical theory provides a visualization which clearly increases comprehension.

Third, (3c) a model is often an ideal. Thus we might say "Jimmy Carter is a model president" (some of us might not say that, but this can serve as a model). Here, as with "scale model," the usage itself informs us of the negative and positive analogies. We are to attend to those characteristics relevant to being a president. We are not saying that all presidents should have large teeth, or a small daughter named Amy (although press coverage would have you believe that one or both of these are indeed relevant). The model is not an ideal in all possible qualities, but in the particular ones which are relevant within a theoretical framework.

Fourth, (3d) a model is transitory. Hans-Georg Gadamer calls a model a "disappearing schema." "The relation to the original [model] that served the painter must be extinguished in the picture" (Gadamer, 1975, p. 128). This is the difference between a model and a portrait. The portrait contains too much of the original. In our terminology, the negative analogy is not made clear. There is a desire to see the real object and not the painting.

> A portrait desires to be understood as a portrait, even when the relation to the original is practically crushed by the actual pictorial content of the picture. This is particularly clear in the case of pictures which are not portraits, but which contain, as one says, elements of portraiture. They too cause one to ask after the original that can be seen behind the picture, and therefore they are more than a mere model which is simply a schema that disappears (Gadamer, 1975, p. 128).

The error here again results from not distinguishing the model from the object being modeled. The painter's model must be simple, unobtrusive. The painting focuses on the role the model plays, not on the model as a person or an individual. The fashion model is similar to us in many ways, being human, but differs consciously and perhaps only in being beautiful.

If we overlook the sexist use of pronouns, Gadamer again is to the point.

> The painter's model is not meant as herself. She serves only to wear a costume or to make gestures clear -- like a dressed-up doll. Contrariwise, someone represented in the portrait is so much himself that he does not appear to be dressed up, even if the splendid costume he is wearing attracts attention (p. 128).

It is not the purpose to show how the typical or standard person would look in a Gucci, but rather the model serves the purpose of showing the Gucci off, of showing how good it can look. The model, in this respect, is an idealization of the actual situation. The model, here again, is similar to what is being modeled (in this case your typical consumer), but is always clearly distinguished and distinguishable from what is being modeled. Failure to do this would be deceptive.

A Brief Historical Survey

Historically, models have fulfilled different needs, but the above primary and secondary characteristics continually reoccur. While what is now described as modeling certainly dates back several centuries (especially in architecture and in the iatrophysics of Harvey and his contempories), the earliest recognition of modeling as a methodological tool is in nineteenth century physics. An equally important, but separate, development is in their use in formal systems in logic and mathematics, and this can be traced to the nineteenth century geometricians. These two separate developments came together in the twentieth century to produce a third, the modern concept of model as it is used in applied mathematics, the social sciences, and cybernetics. These three developments are treated separately.

Physics. James Clerk Maxwell was the first to isolate the analogical relationship in models as a methodological tool. He relates the use of models to other human experiences, and draws a parallel to the world of literature: "The figure of speech or of thought by which we transfer the language and ideas of a familiar science to one with which we are less acquainted may be called Scientific Metaphor" ([1890] II, p. 227).

Maxwell traces earlier uses of analogy back to Kelvin (see, for example, Kelvin, 1842). Kelvin recognized the common use of the inverse square law for both action at a distance and the uniform motion of heat in a homogeneous medium. That is, if we apply the same mathematical formula but alter the nature of the quantities involved, then we shift theories.

The analogy between electricity and heat was noted first by G. S. Ohm in 1827.[1]

> The magnitude of the transition between two adjacent particles, under otherwise exactly similar circumstances, I have assumed as being proportional to the difference of the electric forces existing in the two particles is regarded as proportional to the difference of their temperature (1827, p. 402).

As a result of this analogy, Ohm postulated an electrical quantity (potential) which corresponded to that of temperature in the theory of heat. This is a good example of the role of the neutral analogy. In our terms, it was not clear whether anything corresponded to temperature, but the strength of the positive analogy prompted Ohm to extend the electrical theory to include the notion of a potential. But Ohm, in an important sense, misunderstood the value of his insight, and took the analogy as an indication of a deeper relationship between electricity and heat:

> The form and treatment of the differential equations thus obtained are so similar to those given for the propagation of heat by Fourier and Poisson, that even if there existed no other reasons, we might with perfect justice draw the conclusion that there exists an intimate connexion between both natural phenomena; and this relation of identity increases, the further we pursue it (1827, p. 403).

Thus, although Ohm developed the analogy, and even benefited from it, he did not use the relationship as an analogy, but rather as a possible identity. In our terminology, he confused the model with the prototype.

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[1] This is pointed out by Whittaker (1910, pp. 95-96 and 269). My attention was drawn to this by an extended footnote in Turner (1956, p. 229).

Kelvin was also anticipated by M. Chasles in 1837.[2] Chasles comments in 1840, regarding the inverse square law: "I have shown that this theorem also has an immediate application in the theory of electricity and in that of heat, theories which have a large analogy with that of attraction" (1840, p. 474, my translation).

By the time Maxwell was writing, reasoning by establishing analogies was fairly well established in physics. Maxwell took this a step further by analysing this methodology, and consciously benefiting from this analysis. For Maxwell, Kelvin's analogy is important because it is based on only a partial similarity of the theories.

> It is true, that if we introduce other considerations and observe additional facts, the two subjects will assume very different aspects, but the mathematical resemblance of some of their laws will remain, and may still be made useful in exciting appropriate mathematical ideas (Maxwell, 1855, p. 157).

An analogy exhibits a partial similarity. It depends on establishing a similarity of structure. But it depends equally on establishing a clear separation, so as not to confuse the model with the prototype. The model simplifies, reduces, idealizes, and provides a visualization. It is a temporary explanatory device.

Maxwell then develops two models. The first draws an analogy between the lines of force of electricity and magnetism on the one hand, and curves through space, which indicate, by the direction of the lines, the direction of the force at any assigned point.[3] This is a "geometrical model of the physical phenomenon" (1855, p. 158). The second model uses fine tubes of variable section which carry an incompressible fluid.

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[2] Kelvin's (1842) was reprinted in 1854 in the Philosophical Magazine. At that time he added some notes referring to the anticipations of his own results by M. Chasles and George Green (1828). Green's work did anticipate much of the mathematics. He argued that..."the electric and magnetic fluids are subject to one common law of action, and their theory, considered in a mathematical point of view, consists merely in developing the consequences which flow from this law, modified only by considerations arising from the peculiar constitution of natural bodies with respect to these two kinds of fluid, it is evident the mathematical theory of the latter, must be very intimately connected with that of the former..." (p. 5). But Green does not develop an analogy between the two theories. They simply share a common mathematics.

[3] Green (1828) anticipated this model. However, again it is important to stress that he did not consciously use this as a methodological innovation. Rather, in explaining an equation, the value of a particular variable is "the density of the electricity which would be induced on each of the bodies, in the presence of each other, supposing they all communicated with the earth by means of infinitely thin conducting wires" (p. 36).

> ...since the velocity of the fluid is inversely [sic.] as the section of the tube, we may make the velocity vary according to any given law, by regulating the section of the tube, and in this way we might represent the intensity of the force as well as its direction by the motion of the fluid in these tubes (pp. 158-159).

This extends the use of models by establishing an analogy with an ideal phenomenon, rather than merely noting a similarity between two established theories. The fluid in the tubes is "merely a collection of imaginary properties" (p. 160) and should not be confused with real fluids or even with a hypothetical fluid, i.e. it is not even a theoretical entity.

> By referring everything to the purely geometrical idea of the motion of an imaginary fluid, I hope to attain generality and precision, and to avoid the dangers arising from a premature theory professing to explain the cause of the phenomena. If the results of mere speculation which I have collected are found to be of any use to experimental philosophers, in arranging and interpreting their results, they will have served their purpose, and a mature theory, in which physical facts will be physically explained, will be formed by those who by interrogating Nature herself can obtain the only true solution of the questions which the mathematical theory suggests (p. 159).

The above model is temporary. It is a tentative framework in which to put data into relation with one another. It is a first approximation to be scrapped once a more complete, or otherwise acceptable, account is given.[4]

For Maxwell, the model provides an image or picture of complicated mathematical formulas. It provides a coherent meaning to these formulas, and a way to look at them intuitively. Through the actions of the mechanical model it is possible to actually see (literally) the way in which the algebraic laws manipulate the data. The model is a representation of a physical theory. As an explanatory device, whose importance depends on the prototype (in this case the physical theory which was being modeled) its value is derivative. It is used to provide a coherent picture for the prototype.

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[4] It is important to note that Maxwell maintains a distinction between models and premature theories. In his explanation, however, Maxwell proposes the model in lieu of a theory. As I will argue later (see part II), models are not viable in the absence of any theory, but must be used within theoretical frameworks. In the above, models are used as tentative extensions of an already existing theory, not instead of any theory.

Thus, for Maxwell, the model simplifies the mathematics; it is visualizable; it is an ideal; and it is transitory. Most importantly, while the model captures the structure of the prototype, there is no danger of confusing the model with it. This is not a dispensable attribute of models, to be eliminated if possible, it is an essential ingredient. The explicit dissimilarity between the model and the prototype protects us from confusing the model with what is being modeled. This allows the model to serve an explanatory role, "without making any assumptions as to the physical nature of electricity..." (1855, p. 159). This is argued also, in a more recent discussion, by Gustav Bergmann: "Perhaps the greatest advantage of what I take to be the correct logical description of models is that it keeps one from indulging in a certain kind of dilettante speculation about the 'real nature' of the physical universe" (Bergmann, 1947, p. 486).

Logic and mathematics. The imagery produced by a model, and the consequent realization (again literally) of mathematical formula, developed into one of the central uses of models as formal structures in logic and mathematics. It is interesting to note that the actual term model was not used in this manner until the 1920's. However, nineteenth century geometricians developed "INTERPRETATIONS" or "REALIZATIONS" of axiomatic systems and these are now referred to as "MODEL." Thus, the concept that we now label model had been developed extensively in the 19th century by these geometricians.

Surprisingly, the non-Euclidean geometries were basically ignored by the mathematicians of the time. They were thought of as logical aberrations, made possible by some quirk in the axiomatization of geometry. Even as late as 1902 Henri Poincaŕe could remark, "Most mathematicians regard Lobachevski's geometry only as a mere logical curiosity..." (1902, p. 63). Although Lobachevski and Bolyai argued for the legitimacy of a non-Euclidean geometry, they could not prove its consistency. Morris Kline points to papers found after Bolyai's death in which he continued to try to prove the parallel axiom, thus indicating that Bolyai himself never totally accepted the consistency of non-Euclidean geometry (cf. Kline, 1972, p. 914). In 1854 Riemann suggested that a space of constant curvature in two dimensions could be REALIZED on a surface of a sphere provided that the geodesic on the sphere was taken to be the "straight line." However, he did not explicitly relate this to hyperbolic geometry. This was first done by Eugenio Beltrami in 1868 and then Felix Klein in 1871. They provided INTERPRETATIONS of non-Euclidean geometry within Euclidean geometry. Each statement in non-Euclidean geometry was interpreted as a statement in Euclidean geometry. This provides a relative consistency proof. If Euclidean geometry is consistent, then non-Euclidean geometry is consistent. For if a contradiction was obtainable in the non-Euclidean geometry, then whatever statements corresponded to this in the Euclidean geometry would also be contradictory.

The use of models to establish relative consistency was adapted by G. Peano (1891, pp. 93-94) who developed a method for proving the independence of postulates. If there is a set of postulates, then one is independent of the others if there exists an interpretation, i.e. it is possible to visualize a situation, in which all of the postulates except one are true. This method is probably best known through the work of Hilbert (especially his 1899).

The use of models was of course historically significant by establishing the relative consistency of the non-Euclidean geometries to Euclidean geometry. However, the use of models in consistency proofs can never provide more than relative consistency, since the model itself is always embedded within a theoretical framework which is always, itself, subject to question. Moreover, the adequacy of the translation is always in doubt, since it remains an open question whether appropriate references have been established for the terms of the system, i.e. whether the primitive terms are properly captured by the model and whether the defined terms have appropriate correspondences. These latter problems apply also to the use of models in independence proofs.

A second trend in geometry resulted in the development of models and led in a straightforward way to its use in logic. The inadequacies of Euclid's axioms had been noted by commentators from Pappus and Proclus up through Gauss. Moritz Pasch (1882) argued that these were due to Euclid's reliance on diagrams and intuitions and that the basic notions of point and line must be undefined. In a rigorous development of a system it is essential to eliminate any reference to intuitive concepts. It is useful and legitimate to think of the "meaning" of the terms in working out the proofs in the system, but it must be in no way necessary. Pasch also recognized that if the geometrical terms in a theorem are replaced by others which also satisfy the " basic set " (i.e. premisses), then the deduction goes through as before, but the result is a new theorem.

A rigorous system must contain undefined basic notions. An interpretation of the system provides definitions, but these are always extraneous to the system. The concept of a pure, uninterpreted, formal system is set forth nicely by E. V. Huntington:

> How can we be sure that we do not employ, in our reasoning, some other properties of numbers besides those expressly stated in the axioms? The only way to avoid this danger is to think of our fundamental laws, not as axiomatic propositions about numbers, but as blank forms in which the letters a, b, c, etc. may denote any objects we please and the symbols + and x any rules of combination; such a blank form will become a proposition only when a definite interpretation is given to the letters and symbols--indeed a true proposition for some interpretations and a false proposition for others (Huntington, 1906, p. 2).

In deductions only the blank forms are manipulated, thereby avoiding any "troublesome connotations" which might defile the system. Huntington recognized that this not only allows for a study of the intended subject matter, but may be extended to any system satisfying the postulates. Thus there is a clear distinction between the formal system and the interpretation. However, the intended interpretation still has a heuristic value.

> Any set of consistent postulates might be used as the basis of an abstract deductive theory; but only those sets of postulates are worth studying which are capable of some interesting concrete interpretation (p. 4).

Nevertheless, the distinction between the interpretation and the formal system is enhanced, and the importance of the intended interpretation diminished, by the possibility of multiple associations of models and formal systems. Poincaré argues that no formal system is in a privileged position with regard to any model: "If a phenomena admits of a complete mechanical explanation, it would admit an infinity of others which give equally good accounts of all the particulars revealed by experience" (1890, p. xv). And H. Hertz in his classic (1899) argues, additionally, that no model is in a privileged position with regard to a formal system:

> A system is not completely determined by the fact that it is a model of a given system. An infinite number of systems quite different physically, can be models of one and the same system. Any given system is a model of an infinite number of totally different systems (p. 176).

It is important to note that, whereas Hertz, writing in mechanics uses the term "model," Poincaré, a mathematician, does not. This is particularly odd since many of the researchers at the time, including Poincaré, worked in both fields.

In the 1920's the term model was gradually introduced into logic. It appears in the writings of Carnap, von Neumann, and others. This notion was made precise by Tarski (1935). If L is a set of sentences, and each name (non-logical constant) in L is replaced in all its occurrences with a corresponding variable, the result is a set of sentential functions, L'. "An arbitrary sequence of objects which satisfies every sentential function of the class L' will be called a model or realization of the class L of sentences..." (1935, p. 417). This usage has been extended vastly, but the central aspects remain the same.

In contemporary terminology, we say an interpretation has a domain of objects. This serves as the possible values for the variables. The interpretation also contains a function which, first, assigns objects in the domain to constants in the language, thereby providing referents for the constants. And, second, the function provides for each predicate letter in the language ordered sets of objects from the domain which satisfy the predicate. For example, the function will assign a set of ordered pairs of objects to a two-place predicate letter. A model is an interpretation in which all the axioms are true, that is, it is a "successful" interpretation.

Logical models bring out the primary characteristics of models almost to the complete exclusion of any secondary characteristics. A model must, minimally, preserve structure. This is captured by the algebraic notion of isomorphism, which establishes a formal analogy between two structures.

Isomorphic models preserve the structure of a formal system in the same way. They share the positive analogy, and any differences between them, the negative analogy, must not affect their structure.

Two models of the same system are isomorphic if there is an exact correspondence between objects in their respective domains, and between the relations on these objects. The nature of the particular models, i.e. the kinds of objects and the types of relations, is superfluous to isomorphism, and constitutes the negative analogy. This additional information, however, often provides important heuristic or didactic reason for examining the formal system in the first place. This is a source of additional information which is, strictly, not in the formal system (cf. Kleene, 1950, p. 25). If all models of a formal system are isomorphic, then the system is categorical. In this case, the structure of the formal system is so well defined that all models must share this structure. If this occurs, then the formal system is complete in the sense that it is impossible to add any new, independent, axiom without contradiction.

In summation, as the pressure for more and more formalization accrued, and as the desire for rigour became foremost, the formal system gradually became distinguished from its natural interpretation. Finally, the formal system was regarded as a set of uninterpreted signs, with no meaning for the system. This was generalized so that the link to the natural interpretation was broken, and a formal system could be given an infinite number of interpretations, with none being in a privileged position.

Twentieth Century Developments. As early as the late 1920's various mechanical models were proposed for conditional reflexes in psychology (cf. Boring, 1946, p. 184 for references). But the applicability of this technique to the social sciences has never been completely accepted. Gordon Allport questions the motives of psychologists who are totally committed to the development of mechanical models of human behavior: "The machine model in psychology had its origin not in clinical or social experience, but rather in adulation of the technological success of the physic sciences" (1947, p. 183). Nevertheless, these models set a precedent for the development of mathematical models in the social sciences. In particular, economic models developed in the 1930's and 1940's served as paradigms for the other social sciences.

In applied mathematics models are created to assist in planning and controlling practical situations as diverse as traffic patterns, growth of trees in forests, spread of epidemic diseases, and so on. In these practical situations the model is embedded in a set of operating assumptions which may often be only loosely developed. In the above examples this may range from a set of beliefs about road utilization, or concentration patterns of an illness, or relative advantages of selective versus clear-cut lumbering. But, as in all technological developments, the model utilizes an array of background theories from vastly divergent disciplines, e.g. mechanics, biology, economics, and political science. The relevant variables around which the model is constructed are determined by the operating assumptions as well as the background theories.

The model reduces the complexity of the practical situation by requiring a selection among the possibly relevant factors. Thus, here again, the utility of the model depends on the primary characteristics. While the goal is to produce as close a copy of the actual situation as is possible, there is a deliberate restriction of the situation. The model is not an isomorphism; it does not attempt to replicate the complexity of the actual system under study. The utility of the model depends on its ability to idealize the practical situation by focusing on a few of the possibly infinite number of variables. There is consequently a homomorphism, a many-one mapping, between the actual situation and the model. The variables which are discounted, or ignored, by the model constitute the negative analogy. The effect of these additional factors must be anticipated and evaluated on returning to the actual situation. If these effects are too great, then the utility of the model is limited. On the one hand, it is necessary for the model to function like the real situation, on the other hand, it is necessary to base the model on as few factors as possible.

This use of models to duplicate the results of a natural process is captured, in the extreme, by the cybernetic notion of the "black box" or "closed box" model (cf. Rosenblueth & Weiner, 1945). For the "black box" model, all that is required is that the model function like the prototype. The model specifies a definite relationship between a fixed, finite, number of input variables and a fixed, finite, number of output variables. On a physical level there may be an infinite number of alternative ways in which the inputs produce the outputs, but the model itself specifies none.

The conformity of the input-output relationship is the positive analogy. The negative analogy lay in the total irrelevance of the transformation mechanism. The model itself is simply the functional relationship between sets of inputs and sets of outputs.

Thus, in each of the various historical developments the central characteristics of models remain the same. There is always a partial similarity. There is, of necessity, a positive and a negative analogy. The secondary characteristics, that a model simplifies, visualizes, idealizes, and disappears, are utilized to support the positive analogy and to establish the negative analogy, i.e. to distinguish the model from the prototype.

A Framework for Research

The relationship between a model and a formal theory is one of the open questions in the philosophy of science. In this section I argue that the persistence of this problem is a result of adopting too narrow a view of the structure of scientific research. The accepted position is that theories are hypothetical-deductive systems, and that models are semantic interpretations of these systems. In this respect, the geometric and logical examples discussed earlier serve as paradigms for the analysis of science. The debate started with Duhem (1914) and Campbell (1920), and it persists in the writings of Braithwaite (1953), Nagel (1961), and Hesse (1966). All of these writers accept that theories are deductively organized sets of laws and have a natural interpretation, a model. The controversy is over the extent to which the model is necessary to the theory.

In contrast with this narrow reading of the nature of theories, I propose that theories are complex conceptual schemas which may contain a hypothetical-deductive system and its formal interpretation, but contain much in addition. The use or construction of a model then serves an entirely different purpose than working generally within a theory, or modifying the theory in which the model is embedded. In the second part of this section I examine these divergent purposes, and draw a firm distinction between working within a theory and modeling.

Broadening the Traditional Debate

Pierre Duhem (1914) argued, as did the positivists after him, that models (the natural interpretations) are merely dispensable aids to theory construction, and a subjective method for discovering the structure already present in the theory. Models can therefore be detached and discarded when the theory is fully developed and understood. N. R. Campbell responded,

> ...analogies are not 'aids' to the establishment of theories; they are an utterly essential part of theories, without which theories would be completely valueless and unworthy of the name. It is often suggested that the analogy leads to the formulation of the theory, but that once the theory is formulated the analogy has served its purpose and may be removed and forgotten. Such a suggestion is absolutely false and perniciously misleading (Campbell, 1920, p. 129).

In many respects, Campbell's argument is ahead of his time. The positivists viewed theories as formal hypothetical-deductive systems, independent of the data and of the scientist. Although Campbell accepts the formal hypothetical-deductive structure of a theory, he argues that scientific activity is not limited to the purely formal development of the theory. That is, "If physical science were a purely logical science, if its object were to establish a set of propositions all true and all logically connected but characterized by no other feature, then possibly this view might be correct" (p. 129). But science involves much more. And the role of the researcher cannot be dismissed as simply a necessary evil which should be ignored. The human need to understand science, and the unavoidable fact that humans create science, must be taken into account in any analysis of the structure of science. If we identify science with theory, and we view theories only as formal structures, then we place intrinsic limits on our capacity to understand science. For Campbell, this limitation by itself suffices to have us broaden the explanatory framework to include models.

A formal theory is always, in real situations, an incomplete account. Hesse argues (following Campbell) that a large positive analogy, along with a minimal negative analogy establishes the relevance of a model to a formal theory. The neutral analogy then fills in gaps in the theory, accounts for predictions from the theory, and supplies avenues for future research. While it is clear that all of these features are essential to understanding the activity of science, framing the debate in terms of formal theories and their interpretations unnecessarily restricts this understanding.

Recent developments in the philosophy of science suggest a framework which helps to explicate the nature and roles of models and theories and broadens our perspective on the scientific enterprise even further. Imre Lakatos (1970) argues for a structure which is based on the activity of science. This differs from the previous organization by including the researcher. Lakatos' proposal is that the basic unit for understanding and evaluating science is the methodological research programme. A methodological research programme is a collection of theories organized, on the one hand, around a set of beliefs which are generally not challenged by the researchers in the programme, and, we might say, are accepted for the sake of the argument. These comprise the negative heuristic of the research programme. On the other hand, these theories are organized around certain metaphysical, epistemological, and general research principles and problems. This is the positive heuristic of the research programme. In T. Kuhn's (1970) terminology, this provides a clear "paradigm" for research. From this perspective, a theory provides an explanation of the data which is determined within the programme. Theories are comparable within a programme, and are offered as improvements on previous theories in the programme.

This is a proposed organization of the scientist's activity. It is an account of the logic of discovery. And it is a significant advance over the previously accepted, more restricted, views. In spite of this broadening perspective, for Lakatos theories are hypothetical-deductive systems, and it is necessary to take Lakatos' programme one step further. It is simply not sufficient to subsume the various uses of models in science under the concept of a semantic interpretation of a formal system.[5]

Theories, I suggest, are complex conceptual systems comprising explanations, problems, methods, technological devices, and so on. Theories, in a non-trivial sense, determine the data to be analyzed and structured. Observations are not neutral test items which can be used finally to verify or falsify a theory. As N. R. Hanson (1965) and others have argued, observations themselves are theory-laden. They are ultimately determined by the theory within which they are made.[6]

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[5] While Lakatos does maintain the hypothetical-deductive structure of theories, he too adopts a broader understanding of the nature of models. Within the structure of a research programme, models serve a very specific purpose. "A 'model' is a set of initial conditions (possibly together with some of the observational theories) which one knows is bound to be replaced during the further development of the programme, and one even knows, more or less, how" (1970, p. 136). And, more specifically anticipating our extension: "The positive heuristic sets out a programme which lists a chain of ever more complicated models simulating reality: the scientist's attention is riveted on building his models following instructions which are laid down in the positive part of his programme. He ignores the actual counterexamples, the available 'data'" (p. 135).

[6] See Richards and von Glasersfeld [forthcoming], for a discussion of this, and related, points.

Naive data collection is simply not a fruitful activity. Data is gathered and correlated because it is relevant to something. Observations are made for a purpose, and we construct models to solve problems. But "relevance," "purpose," and "problems" are meaningful only within a theoretical framework. Research, as a result, can only be productive when it is related to a theory. Otherwise there is no reason to do it.

Not inadvertently, this research framework permits a resolution of the Duhem/Campbell controversy. They are both correct -- and yet wrong. As Campbell observed, models are certainly dispensable to theories if we view the latter as restricted to formal systems. But if we view theories as taking in, i.e. as including, the larger context of scientific inquiry, then clearly the use of models is an intrinsic part of that undertaking, even though the particular model is dispensable. For Campbell, if a theory is only a formal structure, then this is an intrinsic limitation of theories, and by itself requires us to broaden the explanatory framework. If a theory is simply a consistent formal, hypothetical-deductive system, then, as he argues, "Any fool can invent a logically satisfactory theory to explain any law" (1920, p. 129).

Modeling and Theories

Theoretical considerations are therefore necessarily prior to building or using a model. We can employ a model, or build a model, only while we are working within a theory, because the theory determines the data the model structures. This is not to suggest that each investigator become a theory-builder, or a philosopher. Rather, it requires that investigators acknowledge the commitments already implicit in their methodology. Experimental research and model building are meaningful activities only insofar as they relate to theories.

This firm dichotomy, between theories and modeling, is not maintained in much of the literature in mathematics education. Even the overt discussions of the uses and function of models in mathematical education conflate the two notions. A.R. Osborne describes three purposes of a model in mathematics education: 1) it is predictive, 2) it is thought stimulating, and 3) it facilitates communication (Osborne, 1976, p. 1). While this is correct, these are also the purposes which a theory in mathematics education would fulfill. And such a characterization neglects the really unique methodological contributions to be gained by working with models.

A model has a particular role to play as a model. This is a different role than that of a theory. In contrast to a theory, a model is not even intended to be complete, or final. It is not intended to capture or explain a situation totally. Whereas a theory must have intrinsic value, a model is useful and worthwhile only to the extent that the prototype has value. The structure and nature of the model depend essentially on the nature of the prototype, and, moreover, it is directed entirely to the needs of the researcher.

A model simplifies, represents, visualizes, preserves structure (mainly), and finally disappears. It is a temporary device used as a possible realization or organization. The model, as a temporary device, permits the consideration of incompatible alternatives, each of which, separately, may be consistent with the theory. As R.B. Braithwaite puts it, "Thinking of scientific theories by means of models is always as-if thinking..." (1953, p. 93). The model, as presented, is subject to revision. Not in the sense that it can be revised, but rather that it is expected to be revised. It is presented as a working hypothesis. There is a partial analogy with the prototype. The investigator's degree of commitment to the model is far less than to the theory. In fact, within this theory it is expected that various divergent models will be proposed. Each may emphasize different aspects of the theory, and thus may be constructed around alternative theoretical principles.

In proposing a model the researcher is suggesting one way in which the theory would organize its data. It is one possible way to do this, but it is not the only way. It is a model, because there is an acknowledged lack of fit both with the data and with the theory. The simplification necessary for research--and empirical confirmation--raises the distinct possibility that we are not in fact testing the theory. We begin by expecting a lack of fit. Thus the criteria of adequacy for the model are entirely different than those for a theory. The limited range of application and the transitory status are just what we gain by utilizing models. We have the luxury of not being committed to the results. And this is a luxury not shared by proponents of theories.

A theory is part of the antecedent conditions for every experiment or prediction. And, as argued by Popper (1934), a false conclusion always gives the option to reject the particular hypothesis (or model) under investigation or the theory as a whole. Whereas this is generally argued to demonstrate the relativity of theories and the impossibility of conclusively supporting a theory, it also indicates the relativity of hypotheses and models. A false prediction may even (rarely) lead to the rejection of a theory and the maintaining of a model. That is, we may attempt to embed the model into a different theory. This requires a translation of the terms of the model into the language and methodology of the new theory. More often this occurs after a new theory is accepted and we are trying to rescue what we can from the remnants of the rejected theory.

Typically, theories are too complex to serve in a practical field. In these situations models are often substituted for theories. This may explain the initial attractiveness of models for researchers in mathematics education. The model used in this way is a simplified version of the theory. The theory itself is the prototype. Generally, the model utilizes several principles central to the theory, and applies them to a particular problem. The model of the theory preserves its basic structure by depending on these principles, but there is a large negative analogy as a result of this simplification. Every effort must be made to make this negative analogy clear, or else predictions based on the model will be useless. It will be impossible to tell whether the bases for the predictions are due to the negative analogy, i.e., to factors in the model not reflected in the prototype, or whether additional factors are obscured because of the negative analogy, i.e., factors in the prototype not reflected in the model.

A model based on theory-prototypes can be used in two ways. First, as a model for research, it creates testable hypotheses. While we never do directly test a theory, we can model a segment of the theory and test the model. Second, as a didactic model, it organizes the subject matter according to selected theoretical principles. When a theory is a prototype there are generally a variety of models proposed which focus on, or emphasize, different aspects of the theory. If one model becomes generally accepted -- wins out -- then the gain in status may be reflected by its incorporation into the theory itself.

The distinction between models and theories, which we have tried to sharpen, is muddied somewhat by a fairly common observation. As R.T. Heimer puts it, "In a very real sense, all scientific theories are models of some segment of reality--as we perceive it" (1967, p. 25). Taken at face value, there is an element of truth in this. However, it still does not abrogate a workable distinction between models and theories. In point of fact its very intelligibility indicates that the distinction is meaningful. A theory determines its own data. It provides a framework for research. There is no prototype, and there is, consequently, no negative or positive analogy.

When viewed from this broader perspective, we can say that, in some respects, theories are, at best, models. They attempt to account for the data which they determine. They try to create patterns and generalizations. In order to do this there is a necessary simplification and a corresponding negative analogy with its own data. But, on the other hand, to call a theory a model is misleading. A model occurs within, or relates to, the framework of a theory. Since the theory itself determines the data there is an inherent confusion between the theory seen as a model and its prototype. The prototype (the data) is being determined by the model (the theory). This interaction between the theory and the data breaks down the model/prototype relationship. It is then not possible to distinguish a prototype which is being modeled by the theory.

The tentative character of models has prompted a similarly misleading suggestion that preliminary versions of theories are models. For example, both Hesse (1967, p. 356) and Apostel (1961, p. 1) refer to the use of models instead of theories to indicate a lower degree of acceptability of the framework. The proposed organization is extremely tentative. Calling it a model underscores its relative uncertainty. A theory "...that is not fully established and to which there are viable alternatives may be called a model" (Hesse, 1967, p. 356). In certain respects this does fit in with our above analysis. The term model is used to point out a positive analogy between the structure of a tentative theory and the data, but more importantly it shows clearly the awareness of a negative analogy. The preliminary theory thus might be considered a model because it has proposed a rough organization which only partly accounts for the data. The claim then is that, at this stage in research, all we can produce is a model. But, for the same reasons presented in the preceding paragraph, a theory, even a tentative theory, is not a model.

Conclusions

The ramifications of this framework for the researcher interested in models is fairly straightforward. First, the researcher is always operating within a tradition (a methodological research programme) and the researcher must be cognizant of this. For mathematics education the programme may be Piagetian, behavorial, logical, linguistic, and so on. Each of these has a fairly well developed metaphysical and epistemological outlook. Each determines data which must be understood, and each delineates general problems to be examined and resolved. Second, within a programme there is a more or less well-accepted, up-to-date, theory which is adopted by the researcher. This provides specific problems and perspectives.

It may be argued, against what is said here, that the confusions between model and theory in mathematics education are due to the immaturity of the field. There has not been time to develop sophisticated theories or unifying explanatory constructs. But, as argued in the previous section, premature theories are different in kind from models. And, from what has been presented in the preceding sections, this is precisely the situation where the dangers are paramount, and the researcher is to be the most particular.

Mathematics education may still be in the stage of generating theories. It is far from being a mature science. Nevertheless, there are still well-defined programmes determining the research. A proposed model functions differently than a theory within these programmes. That is, even if the programme itself is general, or borrowed from other disciplines, a theory within the programme still determines the data and the problems. Models may then be proposed to resolve these problems.

For a specific example, I refer the reader to Steffe, Richards, and von Glasersfeld (1979). The models presented there relate the learning of whole number concepts to the Piagetian tradition, and more specifically, to a theory within this tradition.

The theory and the models are kept separate within this development. This separation is of paramount importance just because the theory is in the process of being changed and continually modified. As a theory matures there is, perhaps, less danger of a confusion.

Philosophically, the issues in building models, as opposed to building theories are straightforward. First, it is necessary to specify exactly what is being modeled, i.e. the prototype. And second, it is necessary to specify the relationship between the model and the prototype by clarifying the positive and negative analogies.

But in practice this is very confusing. In the mathematics education literature there are models for research; models for the child's learning of mathematics; models for the child's construction of mathematics; epistemological, logical, and linguistics models of mathematics itself; teaching models; and so on. And often several of these appear in the same article, as near-synonyms. This results directly from the conflation of theories and models;[7] from the consequent lack of appreciation of the relationship between the model and the prototype; and from the failure to specify the positive and negative analogies.

The use of models as partial analogies -- to simplify, to visualize, to serve as an ideal, and then to disappear -- is an important, if not essential, methodological tool. This is especially true in a field where we must acknowledge that the extreme complexities in the process of learning mathematics preclude definitive answers. However, the significance of a model can only be understood when the negative analogy is clarified. Failure to do this quickly nullifies the advantages of the model. As Braithwaite warns, "The price of the employment of models is eternal vigilance" (1953, p. 93).

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[7] See also the comments on this conflation in Heimer (1967), p. 25.

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EXPERIMENTAL MODELS FOR THE CHILD'S ACQUISITION OF COUNTING AND OF ADDITION AND SUBTRACTION

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In this paper two models, one for the acquisition of counting and one for the acquisition of addition and subtraction, are presented. The models are framed within the general theoretical perspective of constructivism. Constructivism, as we use the term, is an epistemological theory in which acquisition of knowledge is a creative activity of the knowing subject. An immediate implication of this perspective is the active construction of reality by the child. Reality is not "out there" passively awaiting its discovery, rather it is a product of the individual's activity. Acquisition of mathematical knowledge is also a creative activity of the knowing subject. In order to be known it must be constructed by the individual.

The constructive activity of the child is thus the central focus in our investigation. This focus is fundamentally different from the focus of modes of inquiry which emphasize the products of knowing. As Easley (1975) pointed out, the "notion that mathematics at the elementary school level could be a constructive or even a creative discipline is difficult...for most American teachers and educators to accept" (p. 31). Yet, the young child, in its state of mathematical naivete but innate curiosity, bears a deep resemblance to the creative mathematician working on the frontiers of his discipline. For the child also is working on the frontiers of its knowledge.

The elements of the models developed here have, potentially, a wide range of application. We have not, however, investigated many of these possibilities. We wish to underscore the tentative nature of our models and the preliminary status of our investigations. While there is a substantial amount of observational support for our models, we recognize that the models are a result of our interpretation of the observations.

In the first part of this paper, we outline six essential elements of constructivism from a psychological perspective. These elements together suggest the form of the models we elaborate of the child's acquisition of counting and of addition and subtraction. We therefore provide a foundation for our study in both a general theory of epistemology and in psychological principles implied by this theory. In so doing we have attempted to avoid certain pitfalls (identified below) which have marked the long association of psychology and the teaching of arithmetic.

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In the 29th yearbook of the National Society for the Study of Education, F. B. Knight (1930, pp. 145-267) gave an extended treatment of teaching methods based on (at the time) prevailing stimulus-response psychology. In fact, the main psychological basis for teaching arithmetic assumed in the yearbook was Behaviorism (Knight, 1930, p. 5). More than one-fourth of Knight's chapter dealt with drill on basic facts. Wheeler (1935), five years later, outlined twelve main principles of Gestalt psychology and, on that basis, provided hints to mathematics teachers. Buswell (1951), writing 16 years later than Wheeler, outlined the influence of association theories and field theories on the teaching of arithmetic. Buswell's main purpose, however, was to explain the role of meaning theory in the teaching of arithmetic. Buswell was writing at the dawn of the "modern mathematics" movement in the schools but was predated by Brownell (1935) by at least 15 years. Meaning theory was carried on into the "modern mathematics" movement and served as a psychological basis for those programs (Van Engen, 1953; Grossnickle, Brueckner, & Reckzeh, 1968).

There is the danger that curriculum applications of psychological theory distort the theory. Thorndike's (1924) psychological theory of arithmetic is a prime example. As Buswell (1951) pointed out,

> "many students did not read Thorndike with...care. They read his law of exercise and they read his statements regarding the nature of 'bonds.' They then provided the drills that they thought necessary to 'fix' these bonds...For example, Knight's emphasis on mixed drills did not lead to an appreciation of the systematic relationships among number facts" (p. 145).

Conversely, there is the danger that psychological theories will distort the curriculum in application. In fact, this is exactly what happened, according to Buswell (1951), in the case of Wheeler's (1935) application of Gestalt Psychology. Beilin (1976), more recently, recognized that models play an essential role in the relationship of theory and practice when he stated that "What I propose is that an understanding of the processes by which knowledge of mathematics is achieved requires the application of... models" (p. 88).

In the second part of the paper we explicate a three-stage model for acquisition of structural knowledge in mathematics. This three-stage model sprang from the six essential elements of constructivism as well as the more specific models for counting and for addition and subtraction. The model is based on the development and formation of operative structures. At this time we leave open the question of whether it can be adapted to figurative knowledge.

In the third part of the paper we outline a model for the acquisition of counting and in the fourth part we outline a model for acquisition of the inverse relation between addition and subtraction. Much of the data for these models has been accumulated over the past three years. During this period, attempts were made to utilize the principles of constructivism in teaching experiments. This forced a fundamental shift of attention to the activity of the child, and resulted in a vast array of new information which is simply not available when the focus of attention is on the content of the discipline. It seems to us that there are obvious, and extensive, implications for the teaching of counting and addition and subtraction resulting from the models. The work in developing satisfactory models for children's acquisition of the whole number system is far from complete. But, as Osborne (1976) noted, "We have had a rather vacuous fixation on the idea of a model. Educators have been prone to connect boxes and circles with lines and arrows, scatter some labels over the network, and call this apparition a model...Ignoring the hard work that goes into building a model...such people are simply indulging in self aggrandizement" (p. 3).

An essential part of our work involves the incorporation of a model into a theoretical framework which takes into account psychological and epistemological principles. It is only when this broader framework is acknowledged that a perspective can be developed and the significance of the work understood. Piaget (1971) notes, in a similar vein:

> "any advances made by experimental pedagogy, taken as an independent science with regard to its subject matter, must necessarily be linked, as in all sciences, to interdisciplinary researches, if there is to be any question of constituting it as a true science--of its being...not merely descriptive but explanatory" (p. 24).

Elements of Constructivism

> Mathematics is a human activity, it is constructed by people, and any attempt to establish a foundation for mathematics must begin with a thorough investigation into the nature of human knowledge. The foundation for the philosophy of mathematics must be laid in mathematical epistemology.
>
> John Richards
> Epistemology and Mathematical Proof

The seemingly innocuous observation that mathematics is a human activity is often overlooked--in education, in psychology, and in the philosophy of mathematics. Yet it must be taken seriously before we can attempt to understand the nature of mathematics and its acquisition. The key to a fuller understanding of the nature of mathematics lies in broadening the scope of the philosophy of mathematics to take into account the psychology of mathematical creativity. From the constructivist's perspective this requires a deeper understanding of the nature of constructions. Richards' (1977) emphasis on the constructive thought of the individual as it pertains to mathematics is consistent with Piaget's (1970) views concerning the growth of mathematical knowledge. Knowledge is viewed as pertaining to invariances in the living organism's experience rather than to entities, structures, and events in an independently existing world (von Glasersfeld, 1974; Richards & von Glaserfeld, in press). The question of knowledge, then, is cognitive, and psychological theories concerning its construction are of critical importance.

Some essential elements of constructivism from a psychological perspective have been identified by Montangero and Smock (1976), Smock (1976), and von Glasersfeld (1974, 1978). First, a constructivist approach is structural in nature. It takes into account and differentiates developmental structures (e.g., grouping structures) and mathematical structures (e.g., relational structures). It does not focus on superficial levels of behavior (performance), but explicates underlying mental operations. Closely tied in with the structural nature of the constructivist approach is operationalism. Operationalism in this context must be differentiated from the wide-spread, but conceptually narrow, notion of operational definition. Operationalism refers to the functioning of mental structures. It views concepts not as being static entities, but rather as the invariants produced through mental operations (Bridgman, 1936, 1961). Mental operations are part of a total structure, and structure is seen in the organization of operations. Different surface behaviors of a child may be interpreted as springing from the same cognitive structure. For example, conservation of length may be justified by a child through identity or compensatory arguments. Conservation of length per se is interpreted to be the result of infralogical operations.

Second, a constructivist approach emphasizes similarities among seemingly disparate events or fields of knowledge acquisition. Given this emphasis, differences can be described relative to the similarities. Examples taken from Piagetian theory are the similarities between measurement and number which were discovered through analysis of children's behaviors in terms of the development of structures (Sinclair, 1971, p. 153). It was then possible to describe rather precise differences.

Third, the structure of the learning environment must be considered within two frames of reference. On the one hand there are the operational systems controlling the child's experiences and, on the other, there is the content to be learned.[1] This division is particularly important for mathematics education. Usually there are profound differences between the mathematical content to be learned and the existing cognitive structures of the child. Beilin (1971) pointed this out in his observation that "logical relations are not inevitably paralleled by psychological relations...little effort has been expended in testing the relations between the conceptual systems in mathematics and the cognitive systems of the child" (p. 118).

Fourth, the learning of structures in mathematics is stage-like and therefore analogous in character to the development of certain other concepts (Steffe & Smock, 1975). From the constructivist point of view "learning of structures seems to obey the same laws as the natural development of these structures" (Piaget, 1964, p. 17). Learning, indeed, is always a form of reconstruction. Concepts, structures, skills, or anything that is considered "knowledge" cannot be conveyed ready-made from teacher to student or from sender to receiver. They have to be built up, piece by piece, out of elements which must be available to the subject. In the case of mathematical knowledge, the elements involved are not necessarily the same as those which ordinary cognitive development supplies. In fact, one central aspect of our research on model formulation and testing carried out over the past three years concerns this essential point.

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[1] Note that the "content to be learned" is, in fact, the teacher's (i.e., the adult's) standardized system of control for the particular kind of experience. The child's learning, in this sense, is adaptation to an accepted way of operating.

Fifth, the type of learning we are interested in cannot be adequately described as association in a stimulus-response sense. It occurs through assimilation, "The integration of any sort of reality into a structure... which seems to me fundamental in learning, and which seems to me the fundamental relation from the point of view of pedagogical or didactical applications" (Piaget, 1964, p. 18). Assimilation is part of an equilibration process through which knowledge is constructed. Von Glasersfeld (1978) describes assimilation as the adjustment of percepts to conceptual structures and accomodation as the adjustment of concepts to percepts. Accomodation can take the form of creating novel structures or of combining several already assembled structures. Learning is possible only when there is active assimilation and accommodation. Our emphasis, in the constructivist program, is on the activity of the learner. As Piaget (1964) argued, "without this activity there is no possible... pedagogy which significantly transforms the subject" (p. 18). For learning to occur it is necessary to base the more complex structure to be learned on available, simpler structures.

Sixth, two levels of cognitive functioning are recognized--figurative and operative. According to Smock (1976, p. 33) figurations are emphasized by theorists recommending a direct tuition approach to instruction. Figurations include components of perception, speech, imagery, and memory and deal with the perceptually tangible empirical world. In mathematics instruction figurations would include spoken number names, rote counting, point counting, collections of perceptual objects, standard computational algorithms, mathematical symbols, number facts, and perceptual discrimination of geometrical figures. That is to say, figuration depends on perceptual material, either actual or representational (imagery). Operations (in the strict Piagetian sense), on the other hand, are described by abstraction from the actions by means of which the subject compounds his new perceptual experience into larger complexes. Operations are action schemata that produce transformations of states. They act either on representations of events or on their own system. In mathematics, then, operations include unit systems of singulars and pluralities, counting rationally, facets of measurement, and multiplication of whole numbers.

In order to use the above elements of constructivism to gain insights into particular problems, we construct models. The use of the model "provides an accessible means for applying...abstract theories to data..." (Beilin, 1976, p. 73). We expect to find specific relationships suggested by these elements of constructivism among the data of the theory. The use of the model allows us to analyze aspects of the theory and to consider alternative ways in which the elements manifest themselves, for example, in the construction of the whole number system. Models also provide tentative unifying structures which facilitate communication among researchers and between researchers and teachers, and allow for the interpretation of general theoretical propositions in practice.

Two Genevan publications (Sinclair, 1971; Montangero & Smock, 1976) each have stated emphatically that application of Piaget's cognitive development theory to mathematics education must be indirect and requires an intermediate body of research. This requirement can be met through the use of models. In a paper on the nature of models, Richards (1979) stresses the necessity of maintaining a clear separation between a model and the theory within which it is framed. Failure to do so obscures the requirement of an intermediate body of research in application of theory to practice. Moreover it obscures the relationship between the model and what is being modeled -- in this case children's construction of counting and addition and subtraction.

A Model for Acquisition of Structural Knowledge in Mathematics

There are three distinct stages in the model, each stage necessarily preceding the next. The Exploratory (E) stage is the first. Throughout this stage, the mathematical structure under consideration essentially does not exist for the child. Two distinct conditions are possible. In the first case, there is no simpler structure present in the cognition of the child or the simpler structure is far removed from the given structure. A good example is where the total value of a digit (the total value of "4" in 342 is 40) in the tens place of a three digit numeral is to be learned by a child who is preoperational with regard to number (in a Piagetian sense). A mechanism for constructing the meaning of the total value of a digit in the tens place is counting by tens (other than rote counting), which in turn presupposes counting-on by ones. Data (Steffe, in press) show that counting-on by ones assumes number in a Piagetian sense. For a preoperational child, any simpler structure available would be far removed from the total value of a digit.

In the second case, the simpler structure is available and not far removed from the given structure. In this case, the given structure would not demand a reconstruction based on physical action or figurations. Rather, the activities would be designed in such a way that the simpler structure would be modified to form a new structure. In the above example, counting-on by ones represents a simpler structure--a constructive mechanism --not far removed from the total value of a digit in the tens place.

The second stage, Abstraction-Representation (A-R), consists of the reconstruction either of a novel structure at the representational level or of a simpler structure being modified to form a new structure. An example of the latter is where a child who does not know standard computational algorithms is asked to find the sum of 42 and 20. A simpler structure on which the sum can be based is counting-on by tens. However, counting-on by tens has to be modified to incorporate the concept of addition--42 units and 20 units are how many units? The addition situation is assimilated to the simpler structure of counting -on by tens. The simpler structure has to be accommodated to incorporate additive situations, thus becoming more general.

Formalization-Interpretation (F-I) is the third and final stage. At this stage two or more cognitive structures developed at the abstraction-representation stage are integrated to form a new, more inclusive, structure. It is very important to realize that attainment of structural integration or the attainment of a structure is usually manifested by a capacity to acquire a variety of appropriate knowledge.

It is useful to contrast this model for acquisition of structural knowledge with some other work in the field. Polya (1965, p. 133) describes mathematical thinking as growing through three stages--exploration, formalization, and assimilation. Dienes (1971, p. 36) identifies six stages in the growth of mathematical concepts -- free play, games, search for commonalities, representation, symbolization, and formalization. The stages we have identified have similarities to those of Polya and Dienes in that both authors assume an exploration stage. After the exploration stage, however, differences in conceptualization are present. Polya's stages are to be viewed as an application of a phylogenetic principle of teaching--"we should let the human child retrace the great steps of the mental evolution of the human race" (p. 132). Polya's analysis, then, is largely historical. Our model is not based on an historical analysis of the evolution of human thought, but on the ontogenesis of child thought. While there may be analogies between the historical development of mathematics and the ontogenesis of child thought which makes Polya's analysis suggestive, a direct analysis of child thought is more accurate and more productive. Dienes' stages also are developed out of a different framework. They are viewed as being applicable to the learning of axiomatic mathematical structures. As such, the constructs have quite different meanings than those developed in this paper. For example, formalization in Dienes' cycles refers to axiomatization (Dienes, 1971, p. 36). Such a process is so completely foreign to the child's learning of whole number concepts that it is, at best, of dubious value.

We now turn to two specific models. The first is a model for the child's acquisition of counting. The second utilizes aspects of the counting model to produce a model for acquisition of the inverse relation between addition and subtraction. These two models are indicative of a cyclical growth of mathematical concepts.

A Model for Children's Learning of Counting

The model discussed in this section is a model of children's construction of counting by ones. Traditionally, school programs have treated counting as a single type of activity described by the rubric "rational counting" and have considered it unimportant in the formation of mathematical concepts. In fact, children often are discouraged from counting while they are learning numeration and operations with whole numbers. Counting, generally, is viewed as immature mathematical behavior. This dethroning of counting in school mathematics has contributed to what seem to be inappropriate teaching practices, as well as to the current lack of understanding concerning the nature of children's counting and its role in the acquisition of other mathematical knowledge.

Our model of children's learning of counting was developed through intensive teaching experiments with six-, seven-, and eight-year-old children. These experiments have yielded results that are in conflict with traditional school practices. During the teaching experiments, it became obvious that (1) children use counting of some type in acquiring a wide variety of knowledge in arithmetic, and (2) children must learn to count. Children's counting is not adequately described merely by what is generally thought of as rational counting. Various types of counting exist in children's behavior and each type plays a role distinct from the others. The most prominent of these types are rote counting, synchronous counting, point counting, counting with units, cardinal counting, and ordinal counting (Steffe, Hirstein, & Spikes, 1977). It is important to understand these counting types prior to a discussion of the model.

Rote Counting. Rote counting is the most primitive counting behavior observed. Rote counters can utter the first few number names in sequence but have no idea that this sequence of number names can (or should) be uttered synchronous with some other sequence of events. Their counting behavior is isolated and is nothing but a memorized chain of number names. Rote counting provides a language base for other counting types, and, as such, is better considered in the province of language rather than mathematics.

Synchronous counting. The synchronous counter can utter the number names in synchrony with a sequence of events external to the counter. The rote counter must transform his counting behavior into synchronous counting. The new realization is that the sequence of number names can be associated with sequential events other than the uttering of the names.[2] This type of counting is still in the province of language but now has a function beyond imitation.

Point counting. Point counters definitely utter number names in synchrony while pointing to objects in sequence. However this is not sufficient to rate as point counting, because the same object may be pointed to more than once, or objects may be skipped. There has to be an awareness that each object must be pointed to once and only once. That is, point counting must impose an order of precedence on the objects counted (which is not necessary in synchronous counting). The point counter, then, must employ the order relation of precedence in counting activity. But it is not necessary for him to understand position nor does he have to realize that counting a collection of objects in two different ways gives the same number.

Counting then, for the rote, synchronous, or point counter is mainly in the province of language use, and, at most, is tied to the immediate experiential world. Counting at this stage is qualitative, rather than quantitative. Point counters may appear to be quantitative, but teachers usually attribute too much meaning to their counting behavior. Counting for the point counter is basically labeling without inclusion present--the object counted is not thought of as joining the objects already counted.

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[2] These other events may, of course, be internal to the counter, but from an observer's point of view, synchronous counting can be ascertained only in the case of external events.

Counting with units. Counting with units has a symbolic aspect which goes beyond pure point counting. For example, a counter with units could count a collection, part of which was screened from view. The subcollection screened would, by necessity, have to contain few enough objects to allow representation with fingers (or some other convenient means). Given the case where four objects are screened and three visible, the counter with units may extend sequentially four fingers, and then three fingers, while uttering number names. It is such symbolic behavior that is indicative of the presence of arithmetical units. The pure point counter is not capable of this symbolic behavior in counting.

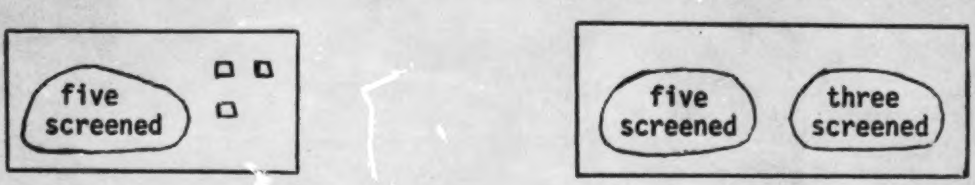
The objects that a point counter tags with number names are units only in a qualitative sense. The objects counted are not fused into a global whole, but are distinct in that they are not individually identical, if for no other reason than that they occupy different spatial positions. The point counter is capable of imposing an order of precedence on the objects through counting. But the order is not necessary for the distinctiveness of the objects. The point counter is well on his way toward construction of arithmetical units in that he has discriminated global aggregates into constituent entities. But the arithmetical unit has not yet been constructed.

Inclusions are not yet present for the counter with units in that an object counted is not viewed as belonging to all those already counted. This is apparent in the counting behavior described above in that the counter did not start with four and count-on the remaining--five, six, seven.

Cardinal counting. Cardinal counting goes beyond counting with units in that it takes into account the inclusive nature of the number sequence. It is a characteristic of cardinal counting that the counter selects an object, says "one," selects another, says "two," selects another, says "three," etc. where "three" does not refer to only the third object, but to the total collection of three. For the counter with units, counting did not carry this meaning. There was little awareness of the significance of the number names being associated with an act of selection beyond synchrony between the acts and the names. The intermediary collections formed through successive selection were not of significance. The cardinal counter can start with, say, the sixth object of a collection and point-count-on the remaining objects.

Ordinal counting. The primary difference between the ordinal counter and the cardinal counter is that the ordinal counter can engage in a double counting process whereas the cardinal counter cannot. Not only can an ordinal counter start with, say, the sixth object in a collection and point-count on the remaining objects, he can count-on while keeping record of those counted--seven is one; eight is two; nine is three; etc.

Each counting type can be divided into two subtypes--counting forward and counting backward. These subtypes are particularly important in cardinal counting and ordinal counting. Counting-on without a running tally and counting-back without a running tally are the counting behaviors corresponding to cardinal counting forward and backward, respectively. Given a collection of five objects screened from view and another visible collection of three as in Figure 1 (a) the cardinal counter can start with five and point-count on the remaining three without tallying. The cardinal counter would be capable also of starting with five in Figure 1(b) and representing the three with fingers and then point-counting on the three to the five. A running tally was not used because the three objects were represented with fingers prior to counting -- not while counting. Counting-back without a running tally is where the child is told there are eight objects in Figure 1(a), and point-counts back to find the number of objects screened--"eight, seven, six; five are screened." The child also could count-back without a running tally in Figure 1(b) if he represented the three objects covered with his fingers prior to counting back.

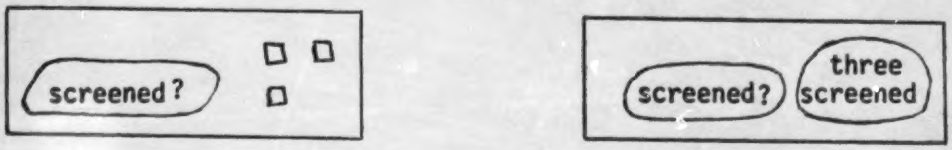


a.

b.

Figure 1. Cardinal Counting Tasks.

The cardinal counter could not solve the tasks represented in Figure 2 (a) or (b) by counting-on or back with a running tally. The tasks, however, undoubtedly would be meaningful for the cardinal counter.



7 Objects
a.

7 Objects
b.

Figure 2. Ordinal Counting Tasks.

To solve task 2 (b) by counting-back with a running tally, the child could count-back: six is one; five is two; four is three--four screened! Or, seven is one; six is two; five is three--four screened! The running tally is generally kept using fingers.

The cardinal counter is aware of position in the sense that the cardinal information that there are five screened can be incorporated into the counting sequence and the counter can start with five in counting. Likewise, in counting back (Figure 1(a)), cardinal information can be obtained from any intermediary point in the counting-back sequence. As the child counted back (Figure 1(a)), and reached "six," awareness existed that the number preceding six was the cardinal number of the objects screened. The cardinal counter, however, is not aware of position in an abstracted sense--positions are not counted as with the ordinal counter. The cardinal counter would not start at six as an initial point in counting, proceed to seven as the second point in counting, etc.

The above distinct types of counting fit nicely into the pattern of our model. The Exploratory stage encompasses rote counting, synchronous counting, and point counting. The main characteristic of this stage is that counting is still in the province of language and is tied to the experiential world. Counting is not yet quantitative. The qualitative nature is shown by the fact that the point counter, the most sophisticated counter in the exploratory stage, does not conserve the numerosity of a collection counted, nor is it understood that if two (or more) collections are in one-to-one correspondence, and one is counted, the number in the other(s) has been determined.

In order to explain how counting becomes quantitative, Piaget's (1952) notion of an arithmetical unit is required. The arithmetical unit carries with it psychological meaning not discussed here. Suffice it to say that it is the hallmark of the number concept which, to Piaget (1952), "is at the same time a class and an asymmetrical relation" (p. 184)[3]. The second stage, termed Abstraction-Representation (A-R), starts with counting units (transitional to cardinal counting) and encompasses cardinal counting and ordinal counting. Children now attribute numerical meaning to counting because they have formed the concept of arithmetical unit. Children may conserve numerosity of collections counted and "solve" problems involving counting and one-to-one correspondence. The achievement of ordinal counting lags behind the achievement of cardinal counting.

The last stage, termed Formalization-Interpretation (F-I), is achieved through integrating counting and reversibility (Steffe, 1973). Counting-on with a running tally is exemplified, as already noted, by a double counting process. For example, to find the number of objects inside a given box, if it is known there are 15 objects in all and 12 objects inside another box, a child may count: "13 is one, 14 is two, 15 is three. There are three objects in the box." A child at the Formalization-Interpretation stage would know that if three would be counted back from 15, 12 would be the result. There would be no need to do this. The child anticipates the result of the potential reverse counting actions. This last stage is achieved through a synthesis of the two-substructures, counting-on and back with a running tally. This is accomplished at the Abstraction-Representation stage. It is a synthesis which is of critical importance in the learning of other content.

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[3] At their first appearance arithmetical units apparently are not assembled into seriated classes, as suggested by Piaget.

A Model for Children's Learning of Addition and Subtraction

A model of children's learning of the inverse relation between addition and subtraction is described in this section. This second example is chosen for two reasons. First, it is heavily influenced by counting. Hence, there are similarities with the foregoing model of how children learn to count. Second, while it is rather straightforward, it demonstrates the difference between the assumptions of current educational practices and the way in which children seem to construct addition and subtraction. Perhaps this difference is illustrated most lucidly by Freudenthal (1973) when he commented that "In the genesis of the number concept, the counting number plays the first and most pregnant role. It should be recognized rather than ignored by developmental psychology and pedagogics" (p. 191). He goes on to advocate strongly that addition and subtraction should be based on counting-on and counting-back, respectively (Freudenthal, 1973, p. 173). However, the foregoing model of children's construction of counting shows that one cannot teach addition and subtraction the way Freudenthal suggests without regard to children's counting abilities. Before meaningful addition and subtraction can be learned, children must construct counting with arithmetical units. Otherwise, addition and subtraction also would be in the province of language and tied to the child's experiental world. It would be nonquantitative.

At some point in the child's development it becomes feasible to base addition and subtraction on counting-on and counting-back. It is indeed essential to do so if the foregoing counting stages are taken into consideration. If our current model of addition and subtraction is correct, one should not expect children to understand the inverse relation between addition and subtraction prior to reaching the final stage in counting. Nor can one expect to view addition as counting-on, or subtraction as counting-back, prior to the second stage of counting. Current practice in school mathematics is to present subtraction as the inverse of addition. While this practice is meant to reduce the amount of factual knowledge the child is expected to learn, it is sound only if the child has already integrated counting-on and counting-back and applied them to addition and subtraction situations.

As in counting, we consider three stages. The Exploratory stage for addition and subtraction is based on counting with arithmetical units. The necessity of including the arithmetical unit in the exploratory stage has been demonstrated by Steffe, Hirstein, and Spikes (1977). Children who understand the arithmetical unit can transform counting with units to cardinal counting and ordinal counting. In such transformations, they shift addition and subtraction to the realm of mental activity.

Counting-back has been observed to lag behind counting-on. Consequently, the Abstraction-Representation stage is based on cardinal counting forward. The Formalization-Interpretation stage is based on the analogous stage for counting. It is the coordination of counting-on with a running tally and counting-back with a running tally that allows the child to construct the inverse relationship between addition and subtraction. In fact, when a child has constructed this inverse relationship, the Formalization-Interpretation stage for addition and subtraction has been entered. This synthesis allows the child to go beyond counting in addition and subtraction.

The notion that the F-I stage is distinct from the two preceding stages receives support in past literature on children's arithmetic. As early as 1927, Judd quoted Thorndike and Stone as advocating that subtraction facts be taught as inverses of addition facts (Judd, 1927, pp. 84-85). Judd perceived correctly that these authors were relying on both the logical and the practical aspects of the distinction between subtraction and addition. Judd observed that "before they have developed the power of abstract thinking, pupils in the earliest grades find themselves confronted with a very elaborate number system..." (p. 86). The "abstract thinking" to which Judd referred meant, in the context of addition, that the view of addition as being derived from counting and grouping experiences does not provide a full psychological account. Counting has to mature into "groups" that may be reassembled, subdivided, or compared. So, the findings that the F-I stage for addition and subtraction represents a higher order psychological state than the A-R stage is not without support in past research on arithmetic.

In the Exploratory stage, the child may find sums and differences using a "counting-all" technique. To find a sum such as 7 + 4, the child may count out seven objects, count out four objects, then point count all of the objects. The child also may use finger counting or other representations of units, but in either case counts-all. Entry into the Exploratory stage for addition and subtraction is characterized by the use of counting-all to process sums or differences.

When the child can count-on without a tally, this approach can be used to do sums. The child is then capable of such techniques as "put 7 in my head and 8, 9, 10. Ten is the answer." where the three objects were point-counted on. If a child has yet to learn counting-back without a tally, finding differences still requires a counting-all technique (when facts are not known automatically). Nevertheless, counting-on without a tally opens up a whole new conception of addition to the child, allowing him to perform much more abstract processes in addition tasks than when he was tied to counting-all techniques. Sums such as 47 + 9 become accessible. In fact, the child can find the sum of any two whole numbers as long as one of the addends can be point-counted on, the other addend is within the numerical experience of the child, and the child can rote count at least as far as the sum. In fact, the Abstraction-Representation stage is characterized by this minimal process for finding sums.

As the child (1) constructs counting-on with a running tally and uses it in solving missing addend sentences such as $15 + \square = 19$, and (2) constructs counting-back with a running tally and uses it in solving subtraction sentences such as $19 - 4 = \square$ the essential structures are being formed for the child to relate addition and subtraction (from solving $19 - 4 = \square$, the child knows the solution to $15 + 4 = \square$). At this point, children display flexible counting techniques in additon and subtraction, consciously making appropriate choices. For example, to solve

$19 - 15 = \boxed{}$, a child might count-on with a running tally, but to solve $19 - 4 = \boxed{}$, he might count-back with a running tally. At the Formalization-Interpretation stage, the child views the addends as units resulting in a new unit: the sum. Joining the units to make the sum can be reversed by taking one of the units from the sum while retaining the other.

In addition to the work of Steffe, Hirstein, and Spikes (1977) and Hirstein (1978), there is other empirical evidence that children utilize counting in addition and subtraction. One of the earliest studies concerning children's strategies in finding the sum of two whole numbers was reported by Brownell (1928). He asserted that "...the way in which (children) learn the (simple addition) combinations is determined most largely by the degree of maturity attained in thinking about concrete numbers" (pp. 226-227). Concrete numbers, to Brownell, incorporate apprehension of visual stimuli (dots arranged in particular geometrical configurations). The first step he identified in apprehension of concrete number was counting the individual dots of the configuration one by one. The next step was partial counting. For example (p. 222), a child would apprend $\begin{smallmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \end{smallmatrix}$ by apprehending the first four dots as "4" and then counting "4--5, 6, 7." Grouping was the next step. In the example, a child would not count the three dots on to the four, but rather would think "there is 4, and here is 3 more; that makes 7" (p. 222). Brownell viewed grouping as an advance from partial counting toward more abstract methods of apprehending concrete numbers. In grouping, the dots are treated not as separate items but rather as combined into unified wholes.

The last step Brownell identified was called "multiplication and conversion." Here, the essential characteristic is that children impose upon a concrete representation of number a pattern of thinking in abstract terms. Multiplication refers to apprehension of equal groups of items through counting by threes, fours, or fives (in Brownell's study). Conversion refers to apprehension of $\begin{smallmatrix}\cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot\\ \cdot\,\cdot & \cdot\,\cdot & \cdot\end{smallmatrix}$, for example, as 12 minus 1, etc. The term "conversion" was applied because the child apparently converts the visual display into abstract symbols.

Brownell claimed that the way children learned the additive combinations was determined almost exclusively by the way they apprehended concrete number. The difference was that the children he studied rarely used simple counting with addition combinations and, of course, did not habituate concrete number. In one example he gave, the child tried to memorize addition combinations without a more mature apprehension of concrete number beyond simple counting. The child had no method of testing the correctness of the answer nor of generating the sum if it could not be recalled. The demands on memory became excessive, and facts became confused. In another example, the child used partial counting to find sums, and in yet another, the child used grouping to find sums (4 + 3 became: 4 and 2 is 6, and one more is 7). Brownell's study demonstrated the existence of (1) partial counting, which we call counting-on without a tally, and (2) addition beyond counting. Moreover, Brownell clearly believed that partial counting was a more mature form of counting than what he called simple counting.

Conclusions

The stages identified in this paper are applicable only to the particular concepts studied with regard to the whole numbers. Their generality remains to be established. Whether they are, for example, general enough to incorporate multiplication or fractions is yet to be investigated. But because the general psychological construct of equilibration offered a theoretical framework for stages, their potential for characterizing a significant portion of all that is mathematical learning and teaching is exciting. They are especially significant when considered in view of Freudenthal's (1973, p. 128) levels of the learning process. On the bottom level, Freudenthal describes the child as operating on "the concepts by manipulation, but he does not know what he is doing" (p. 128). The second level is described as a level of reflection of what was done on the bottom level--"the organizational means of the bottom level would be the objects of analysis..." (p. 128). On this second or higher level, Freudenthal describes the subject matter under consideration as being put into a "linguistic pattern" (p. 123). These levels are analogous to stages described in this paper, but as Freudenthal states "What those levels are, I can only locally explain."

The constructivist model for acquisition of structural knowledge in mathematics and the two particular models for counting and addition and subtraction provide a psychological and epistemological foundation both for future research and for teaching practices. The pedagogical approaches implicit in the models are in sharp contrast with generally accepted practice. Perusal of current commercial primary textbooks in mathematics (and mathematics programs available through other sources) reveal that the program writers, in the main, develop their content sequences through logical analysis ignoring the child's constructions. From the constructivist perspective, the conceptual steps involved in the development of a certain procedure must be integrated with the child's developing conceptual structures. A good example of the failure to take this into account is the procedure of finding the sum of, say, 8 and 6 by finding the sum of 10 and 4. The associative property of addition is used as a rationale for the procedure. As such, there is no attempt to integrate the procedure with counting skills of the child. The associative property does not begin to explain the dynamics of the procedure even if the child knows there is a reason for transforming the given sum (8 + 6) to an equal sum (10 + 4). Moreover, even programs which purportedly are based in cognitive development generally seem to ignore the inherent epistemological problem that is addressed by constructivism. In short, textbook authors produce their instructional analyses in mathematics and instructional technology, largely ignoring the child's own conceputal construction of the mathematics. The meaning theory utilized by the textbook authors is syntactical and semantical (Van Engen, 1949). Such meaning theory springs from an adult's understanding of mathematics and an adult's interpretation of the mathematics -- not from a study of what children can and cannot do.

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A LOGICAL-COGNITIVE VIEW OF LEARNING AND TEACHING MATHEMATICAL CONCEPTS

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The purpose of this paper is to synthesize three constructivistic learning theories into a point of view useful for learning and teaching mathematical concepts. We assume that learning mathematical concepts depends upon the nature of the learner, the nature of mathematics, and the nature of instruction (Beilin, 1976). In particular, we make the assumption that learning mathematical concepts follows a constructivist epistemology of learning and teaching (Steffe & Hatfield, Note 1); that is, conceptual structures are spontaneously and gradually constructured through the learner's active participation in abstraction and generalization processes -- actions are susceptible to repetition and then to generalization. The resulting conceptual structures then reorganize themselves through reflective processes. This construction of new structures is motivated in part by the need to resolve differences between what is known from the existing structures and what is newly experienced. In short, the proposed logical-cognitive point of view is in effect a synthesis of the theories of Dienes, Piaget, and Skemp as they apply to mathematical concept acquisition, though the interpretations are those of the present authors. Our paper can be viewed as providing orientation and direction for constructing specific models of learning and teaching mathematical concepts.

This paper is divided into four major parts. In the first part we discuss the psychological foundation on which this view is based, which is an interpretation of Piaget's developmental psychology. In the second part the structure of mathematical concepts from both mathematical and psychological points of view are discussed. This part is based principally on Skemp's (1971, chaps. 2, 3, & 4) work. In the third part the assumption is made that mathematical instruction depends largely on how students learn mathematics and then instructional considerations in terms of consequences of the previous psychological and mathematical considerations are discussed. Instructional considerations are based primarily on the work of Dienes (1960, 1964, 1971) and Skemp (1971). In the fourth part implications and suggested research are found.

Psychological Considerations

The vast Piagetian literature will not be reviewed or outlined here, but some particularly relevant portions of Piaget's psychology will be highlighted so that the form of the model can be more easily put into perspective. More specifically, an emphasis will be placed on Piaget's periods of mental development and the abstraction processes which accompany these. Our discussion is predicated on the idea that similarities exist between cognitive development and individual concept acquisition.

Three Periods of Mental Development

Piaget and Inhelder (1969) view the mental development of the child as a succession of three periods.

> Each of these extends the preceding period, reconstructs it on a new level, and later surpasses it to an even greater degree. This is true even of the first period, for the evolution of the sensori-motor schemes extends and surpasses the evolution of the organic structures which takes place during embryogenesis. Semiotic relations, thought, and interpersonal connections internalize these schemes of actions by reconstructing them on a new level of representation, and surpass them until all new concrete operations and cooperative structures have been established. Finally, after the age of eleven or twelve, nascent formal thought restructures the concrete operations by subordinating them to new structures whose development will continue throughout adolescence and all of later life (along with many other transformations as well) (pp. 152-153).

As the child passes through these three periods, he constitutes his knowledge as a system of transformations which becomes progressively more sophisticated.

First period. Cognitive structures have their origins in actions. In this section, we will emphasize only the types of actions observed by Piaget (1971a) in his description of the first period and then we will discuss their relations to abstracting processes later in the paper.

Piaget distinguishes between two types of action. First, there are individual actions such as throwing, pushing, touching, and rubbing. Second, there are coordinated actions such as joining together, succeeding each other in a temporal order, setting up a correspondence between one action and another, and establishing intersections among actions. "For... the distinction between individual actions and coordinated ones is only a gradual and not a sharply discontinuous one. Even pushing, touching, or rubbing has a simple type of organization of small subactions" (Piaget, 1971a, p. 19). These two types of actions not only form the basis of Piaget's logical structures, but they also are the basis for the taxonomy of concepts and the learning processes to be discussed later.

Second period. According to Piaget and Inhelder (1969), during the second period two successive levels between action and thought develop. First, the actions are internalized as mental representations or images (sometimes referred to as the preoperational stage occurring generally from the ages of 2 to 7). Second, "After seven or eight there is the level of operations, which concern transformations of reality by means of internalized actions that are grouped into coherent, reversible systems (joining and separating, etc.)" (p. 93). Thus the second period involves representation and concrete operational levels of thought.

The concrete operational level is distinct from the representation level in that it deals with operations instead of images or static states. At this level and beyond: (1) static situations are subordinated to transformations in that every state is conceived of as the result of a transformation, (2) transformations are assimilated to operations, and (3) concrete operational thought is characterized by an extension of the actual in the direction of the potential. At this point of development the operations are called concrete because they are still related directly to objects rather than to hypotheses; that is, concrete thought remains essentially attached to empirical reality. Furthermore, the system of concrete operations can handle only a limited set of potential transformations.

Third period. The emergence of the last period is marked by a reversal of the direction of thinking between reality and possibility (hypotheses) (Inhelder & Piaget, 1958). Instead of building internal structures from empirical data, formal thought begins with the formation of hypotheses which are later empirically verified. Piaget refers to this formal thinking as hypothetico-deductive; that is, deduction no longer refers directly to perceived realities but to hypothetical statements.

These newly acquired thought processes result from both a structural integration and combinatorial system. Structural integration relates to various cognitive consolidations of previously "unrelated" schemes. For example, inversion and reciprocity are necessary conditions of equilibrium for direct and relational concrete operations respectively, but no general synthesis of the two forms of reversibility occur during the concrete operational stage. However, formal structures (i.e.,the structures of the third period) bring together both inversions and reciprocities in a single system of transformations. In particular, this structural integration assists subjects in successfully solving proportion problems. Thus, for example, adding weights on one side of a balance beam can be compensated for by increasing the length on the other side.

The most general property characterizing formal thought is a combinatorial system whereby the subject is capable of logically determining all possible outcomes of an experiment. Thus, operations are no longer necessarily tied to objects or reality because the subject can logically derive all possibilities from stated hypotheses. The possibilities can be thought of as a system of second-order operations in the sense that they describe relations between relations (hypotheses) rather than referring to objects directly (concrete or first-order operations). This idea of reflecting on mental constructs will be discussed again when dealing with the third learning-instructional phase.

Abstraction Processes

In this section two abstraction processes and equilibration will be examined as factors contributing to cognitive growth. The two abstraction processes are called simple abstraction and reflective abstraction (Piaget, 1971a). Simple abstraction is the process by which a learner abstracts what is common among the properties that members of a certain class of objects possess. For example, trapezoid is the abstraction of common properties possessed by all trapezoids, namely, quadrilaterals with at least two opposite sides parallel. The product of simple abstraction is a figural concept. Reflective abstraction is the process by which a learner abstracts what is common among the properties that the members of a certain class of actions between objects possess. It is the action between objects that is abstracted not the properties of the objects being acted upon. For example, addition of whole numbers involves the abstraction of additive coordinations (the coordinated action of joining sets together) and sequential coordinations (counting). The objects comprising the various sets to be joined are not abstracted as part of the addition scheme. For another example, the sine function is a coordinated action of assigning outputs to inputs. In the unit circle interpretation, this is the action of assigning ordinates to angle measure. The product of reflective abstraction is an operational concept. It is Piaget's contention that simple and reflective abstractions are radically different and, in fact, this difference is one of the keys to understanding cognitive growth.[1]

Reflective abstraction is used in two principal senses (1977a):

> In a first psychological sense abstraction is the transposition from one hierarchical level to another level (for instance, from the level of action to the level of operation). In a second psychological sense reflection refers to the mental process of reflection, that is, at the level of thought a reorganization takes place (pp. 17-18).

The two kinds of reflective abstraction are similar in that they both involve reflecting upon actions; in one case the reflection is external, i.e., reflecting upon actions between objects; and in the other case it is internal, i.e., reflecting upon the operations.[2] Reflective abstraction, in the first sense, has as its product operational concepts; in the second sense, it has as its product reorganization of cognitive structures. These products are intrinsically different as well as existing on different levels. Structural reorganization is a realignment of knowledge (figural and operational concepts) so that the relationships of one part to another are changed. This is accomplished by attending to and organizing the relationships between concepts into some new scheme. In contrast, operational concepts are internalized actions which are assimilated by attending to and organizing actions between objects.

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[1] Piaget also uses the terms physical experience and logico-mathematical experience to address the role of exercise and of acquired experience in the actions performed upon objects. According to Piaget and Inhelder (1969) physical experience consists of acting upon objects in order to abstract their properties, whereas logico-mathematical experience consists of acting upon objects with a view to learning the result of the coordination of the actions. We would use the terms physical experience and simple abstraction as well as logico-mathematical experience and reflective abstraction (of the first sense) interchangeably. Our choice of terms was based on common usage by other authors.

[2] Skemp (1971, chap. 4) gives an account of reflective abstraction as described in Piaget's second sense. Further elaboration on this theme will be discussed in the section under instructional considerations.

There are similarities between the reflective abstraction processes (in both senses) and the double aspects of cognitive growth that Piaget refers to as assimilation and accommodation. Reflective abstractions (first sense) and Piaget's assimilation process both involve the integration of external elements into evolving or completed conceptual structures; and reflective abstraction (second sense) and accommodation involve modification of a structure by the elements it assimilates. In other words, we point out the analogy that just as cognitive adaptation consists of an equilibrium between assimilation and accommodation in Piagetian theory, the growth of mathematical structures depends on an equilibrium between the two senses of reflective abstraction. "Equilibration" tempers the assimilative and accommodative aspects of abstraction to maintain a balance between no change and random change.

The important point is that reality acts upon the mind and the mind acts upon reality, each element being modified by the other. By the reflective abstraction processes, the schemes assimilate reality but in that very process are themselves modified. The mind structures reality and in that process the mind is itself restructured--there is an interaction between reality and the learner that is much more sophisticated than the one-dimensional interaction between stimulus and response hypothesized by an associationist model.

Before this particular interpretation of Piaget's psychology of cognitive growth can be applied to mathematics instruction, it is necessary to consider mathematics itself. What is a mathematical concept? It is this issue that we address in the next section.

Mathematical Considerations

It is essential that the meaning of "concept" be resolutely clear in the course of our discussions, especially since our proposed point of view is applied to mathematical concepts. Although the word "concept" has broad meaning in mathematics education literature and general discourses, some researchers have given the term more precise meaning. For example, Shumway (1971) defines concept within a mathematical context.

> A concept is a partitioning of a class X into two disjoint classes X_1 and X_2. The elements of the class X_1 are called positive instances of the concept and elements of the class X_2 are called negative instances of the concept. The class X is called the universal class over which the concept is defined. The notion of partition requires that the class X be the union of the class X_1 and the class X_2 and that the classes X_1 and X_2 are disjoint. To say that a student knows the concept over the class X is to say that given any object from the class X the student is able to identify the object as a member of the class X_1 or the class X_2 associated with the concept over the class X (p. 218).

This definition is particularly appropriate for "class" concepts such as those used with the classical experiments reported in A Study of Thinking (Bruner, Goodnow, & Austin, 1956). For example, with two attributes, say shape (square, circle) and color (blue, red), we may define a concept called "blarp" to be the class of blue squares. These contrived concepts have their role in basic research but may be difficult to apply directly to the classroom environment; and furthermore they usually related just to figural concepts. We not only want to discuss mathematical concepts per se, but we also wish to study classes of actions as well as classes of objects. Our rationale for the taxonomy to be proposed is based on the distinction between objects and coordinated actions as suggested by Piagetian developmental psychology.

By objects we have in mind such mathematical instances of polygons, whole numbers, and polynomials. Although these concepts are mathematical abstractions, and as such are not physical in the literal sense of being susceptible to touch, the corresponding instances can be considered as things or objects to be dealt with or manipulated. On the other hand, by coordinated actions we have in mind instances of mathematical concepts such as congruence of polygons, addition of whole numbers, and multiplication of polynomials. These specific concepts are abstractions of well-defined actions performed on other concepts (objects in these cases). Sometimes it is helpful to think of analogous classifications or designations for objects and coordinated actions, although this approach may tend to oversimplify the distinctions. One analogous designation is suggested by static states versus actions. For another, the linguistic distinction of nouns and verbs corresponds in general to the conceptual distinction of figurations and of operations. Further distinctions between objects and coordinated actions will be discussed throughout the paper. Indeed, much of this proposed point of view rests on these distinctions.

Figural Concepts

The first category of our taxonomy is figural concepts. They are considered to be abstractions resulting from the learner abstracting what is in common among the static properties that instances of a class possess. In more general terms, it is the result of drawing from a number of different situations something which is common to them all (Dienes, 1961). Using the language of the previous section, a figural concept is the product of simple abstraction.

Operational Concepts

The second category of this taxonomy is operational concepts. They are considered to be abstractions of coordinated actions as applied to various figural concepts or other operational concepts. Operational concepts are products of reflective abstraction. This category can be further subdivided into relations, functions, and higher-order operations. Although they each have distinctive characteristics, they are sufficiently alike from a psychological point of view to be classified into one category (i.e., they each involve coordinated actions).

Relations. Examples of relations are the order and equivalence relations (less than and equality, e.g.) and even more simply the relation of the diameter and circumference of a circle by the constant π (in symbols, $\frac{C}{D} = \pi$). In general, relations involve actions which associate figural concepts by providing a connecting link between their static states. The essence of an individual relation from a psychological point of view is the coordinated action, or relationship, or association between the corresponding concepts. Contrary to Henderson's (1970, p. 173) taxonomy of concepts whereby he classifies concepts according to their functional context, we classify concepts as being figural or operational depending on the characteristics of the elements forming the class of instances.[3] Therefore, a relation is an operational concept because it requires that the class of coordinated actions relating ordered pair components be abstracted.

Functions. Functions are special types of relations which can be described as actions which assign to each figural concept (usually its measure) exactly one other figural concept (usually its measure) by some well-defined assignment. Although functions are relations from both mathematical and psychological points of view, they differ from relations in that they impart an "unique-assigning" action in a specified direction. These distinctions of specified direction and unique assignment are quite apparent with binary functions which assign to any two figural concepts another figural concept according to a definition of the respective binary function. Ordinary addition and multiplication on whole numbers are examples of common binary functions. Functions play a catalytic role in the sense of imparting directed action between concepts within a mathematical structure.

Higher-order operations. Higher-order operations are laws of properties which determine the type of structure by imposing a sort of "control" over how relations and functions interact. In more general terms, higher-order operations can be described as relations or functions between other relations or functions. Reflexivity, symmetry, and transitivity are examples of higher-order operations associated with order and equivalence relations. The presence or absence of various combinations of these higher-order operations determine the nature of structures. For example, some higher-order operations pertaining to binary functions are closure, commutativity, associativity, and distributivity. Within the structure of whole numbers, for example, addition not only possesses the properties of closure, commutativity, and associativity, but addition together with multiplication are related by the distributive property.

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[3] Henderson defined a functional taxonomy of concepts whereby he classified every concept as either denotative, nondenotative, or attributive. It should be noted that in Henderson's taxonomy a single concept can be classified as either denotative or attributive. He gives the example, "If congruence is conceived as a relation, that is, a set or order of pairs determined by 'x is congruent to y,' the concept is denotative. If it is conceived as the property of all pairs of objects x and y such that x is congruent to y, it is attributive." Therefore, in Henderson's taxonomy, being classified as denotative or attributive is a function of how the concept is used by the teacher. Skemp (1977) alludes to a similar distinction between a relation and a relationship.

In summing up our taxonomy of concepts, there exists a hierarchical type structure beginning with figural concepts on which a conceptual structure will be formed. Figural concepts are then connected by relations and functions that permit the skeletal formation of structures. In particular, functions impart directed actions to these structures by assigning to figural concepts other figural concepts and finally, relations and functions are related and governed by certain higher-order operations which generally provide a sense of order and organization to our structures. These conceptual structures are both psychological and mathematical. Conceptual structures are psychological in the sense that they are internal structures that are continually added to, modified, and consolidated; they are in themselves mental tools to apply to future learning tasks. Conceptual structures are mathematical in the sense that they possess logical organization and sometimes form recognizable mathematical structures such as semi-groups, groups, rings, fields, networks, and topologies. From both psychological and mathematical viewpoints there are endless ways in which figural concepts can be interrelated with various relations, functions, and higher-order operations. The study of the ways in which these conceptual structures are formed and function is assumed to be the very essence of the psychology of learning mathematics (Skemp, 1971).

Psychological and Mathematical Comparisons

Piaget's operational structures of intelligence and our previously discussed conceptual structures share certain similarities.[4] From the standpoint of acquisition and form, both are internalized structures acquired by abstraction and self-regulated by equilibration, and both are mental constructs which serve as mental tools for further acquisition. There are also similarities from the standpoint of content or structure. Figural concepts relate to Piaget's physical knowledge (simple abstractions from the static properties of objects). Weight, for example, is a static characteristic of an object and thus would be classified by Piaget as physical knowledge. In our mathematical structures weight would be classified as a figural concept. Relations, functions, and higher-order relations are analogous to Piaget's logico-mathematical knowledge (reflective abstractions of coordinated actions which relate or transform objects or states). For instance, addition is considered to be an operation in both types of structures.

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[4] The words "scheme" and "schema" have different meanings to Piaget and Skemp. Piaget (1973, p. 66) calls schemes of an action the general structure of this action, conserving itself during these repetitions, consolidating itself by exercise, and applying itself to situations which vary because of modifications of milieu. However, images in themselves are not schemes so Piaget uses the term schema to designate them. In short, a schema is a simplified image, whereas a scheme represents what can be repeated and generalized in an action (Piaget, 1970, p. 719). On the other hand, Skemp uses schema as the general psychological term for a mental structure. We compromised and used the more general designation of conceptual structure to represent mental structures comprised of mathematical concepts.

Operations are the essence of Piaget's cognitive structures and we believe that functions, or more generally operational concepts, play a dominant role within mathematical structures. In other words, abstracted coordinated actions play the major role in both systems of structures. For example, the essence of algebra is not the class of polynomial expressions, but rather the various ways these expressions are related (equations, inequalities, e.g.) and acted upon (addition, multiplication, e.g.).

For the present discussion and purposes of this paper, it was not felt appropriate to discuss cognitive structures in the detail suggested by such mathematical terminology as "groupings" or 4-groups (Inhelder & Piaget, 1958). However, we wish to strengthen our position by reminding the reader that Piaget (1971b) explicitly mentioned a close correspondence between mathematical structures and the fundamental operational structures of thought when he asserted, "Yet mathematics constitutes a direct extension of logic itself, so much so that it is actually impossible to draw a firm line of demarcation between these two fields" (p. 44). In particular, Piaget stated, "As early as the 'concrete operations'...we find algebraic structures in the logical 'groupings' of classes, ordering structures in the 'groupings' of relationships, and topological structures in the child's spontaneous form of geometry...order" (p. 46).

Finally, a distinction between mathematical (conceptual) and operational structures along the dimension of consciousness is mentioned. Conceptual structures are conscious within a subject's mind in that one is required to reflect consciously on mathematical structures by means of a technical language comprising a very particular form of symbolism, and demanding a greater or lesser degree of abstraction (Piaget, 1971b). On the other hand the operational structures direct the subject's reasoning but are not conscious structures, that is, they do not constitute objects of reflection.

Instructional Considerations

Instruction involves bringing learner and mathematics together in such a manner that understanding takes place. According to Piaget (1971a, p. 15), "To know is to assimilate reality into systems of transformations." Skemp (1971) describes understanding this way: "To understand something means to assimilate it into an appropriate schema" (p. 46). Therefore, it would appear that the teacher's role in the instruction and learning processes is to prepare mathematics for abstraction into the learner's conceptual structures in a manner supportive of the learner's abstracting capabilities. The result should be conceptual learning; that is, a building up of internal conceptual structures which consolidate within themselves and assist with future learning.

We assume that conceptual structures follow the same basic formation patterns and abstraction processes as operational structures, so that

Piaget's three periods will form broad guidelines for describing phases and processes associated with concept acquisition. In addition to Piaget's developmental psychology, our conceptualization of learning-instruction phases has been influenced by the work of Steffe and Smock (1975), Beilin (1975), Dienes (1961, 1971), and Skemp (1971). Our three learning-instructional phases will be referred to as (1) exploration, (2) assimilation, and (3) formalization[5].

The Exploration Phase

The purpose of this initial phase is to begin the abstraction processes by presenting to the learner instances of the impending concept. The choice of instances are determined by several instructional factors, among which are: interpretation, mode of representation, embodiment, and attribute variability.

The interpretation factor pertains to the mathematical approach or model used to embody a concept. In teaching the sine and cosine functions, for example, the unit circle or right triangle interpretations can be used as models of the functions. In teaching rational numbers the goal may be to eventually synthesize the fractional parts, division, and ratio interpretations. Over a period of time, multiple interpretations would appear to strengthen a learner's understanding of a concept by providing more interlocking bonds within appropriate conceptual structures as well as improving the likelihood of transfer and application.

The mode of representation factor pertains to the form in which mathematics is communicated to the learner. We will identify four modes and refer to them as: concrete, visual, algebraic-symbolic and verbal. The concrete mode refers to hands-on type of materials whereas the visual mode refers to diagrams, pictures, figures and so forth which are usually drawn on paper or a blackboard in such a manner that the visual "resembles" the concept (it is in effect a pictograph representing an individual instance of the concept). The algebraic-symbolic (or just symbolic) mode refers to numerals, letters and various combinations together with other special symbols such as $f(x)$, $\int_a^b f(x)dx$, $a|b$, $x^2 + x + 3$, and $\sqrt{4}$. Some symbolic representations are also partly visual such as $a|b$, $a||b$, and ΔABC.

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[5] Polya (1963) called his three consecutive learning phases exploration, formalization, and assimilation; and briefly describes the learning process: "Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits" (p. 608). Our choice of words for the alleged learning-instructional phases were "derived" mostly from Steffe and Smock's model. The fact that our descriptors are the same as Polya's (although not necessarily their meaning) appears to be a coincidence.

The verbal mode refers basically to the written word but also includes oral speech. Word problems are a classic example of the verbal modality. Although each mode is distinct, they are frequently used together as, for example, when triangles (visual) are labeled (algebraic-symbolic).

The embodiment factor pertains to the actual manipulatives or visuals used to embody the chosen interpretation. For example, commonly used embodiments for the fractional parts interpretation of rational number are: Cuisenaire rods, fractional bars, fractional tiles, number lines, sets, rectangles, and "pies." In short, the interpretation factor refers to the mathematical setting or model, whereas the embodiment factor refers to the actual hands-on devices used within appropriate interpretations. By the very definition of embodiment, it is applicable only to the concrete and visual modes of representation. According to Dienes (1964) multiple embodiments increase the probability that a concept will be a "true" abstraction rather than an abstraction of one certain embodiment. The fractional parts interpretation of fractions, for example, certainly has wider application than to only "pies."

The attribute variability factor pertains to variance between and within irrelevant attributes (noise) possessed by each instance of a concept. Usually the initial instances contain little irrelevant variation so that the concept is reasonably clear, and then later in the abstraction process more noise is introduced in order to more clearly see the boundary or extent of the concept. More specifically the choice of instance attributes in designing instructional sequences depends to a great extent on the distinction between figural and operational concepts, since the learning processes associated with their acquisitions appear to be more unlike than alike. The common characteristics of object-instances belonging to a figural concept are abstracted by experiencing instances varying in the presence or absence of attributes as well as variance within an attribute. If all the defining attributes are present in an instance it is called a positive instance; that is, it belongs to the class of all instances comprising the concept. Positive instances vary between and within irrelevant attributes. For example, in abstracting the concept of rectangles, all angles being right angles and opposite sides being congruent are invariant (defining) attributes; whereas size, relative position, color, and material of instances are irrelevant attributes, and must be permitted to vary.[6]

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[6] Our attribute variability factor appears to be similar to Dienes' (1971, p. 31) mathematical variability principle.

> A mathematical concept usually contains a certain number of variables and it is the constancy of the relationship between these, while the variables themselves vary, that constitutes the mathematical concept.... For example, with the concept of a parallelogram we can vary the shape by varying the angles and the lengths of the opposite sides; we can vary the position, as long as we keep the opposite sides parallel. Clearly a set of congruent parallelograms placed in the same position would not be a suitable set of experiences for the growth of the concept (p. 29).

If the defining attributes are not present in an instance, it is called a negative instance. In sum, qualitative variation (varying attributes from instance to instance) is important for figural concept acquisition. On the other hand, the common patterns of action-instances belonging to an operational concept are abstracted by experiencing instances varying in the presence or absence of attributes as well as in the numbers used to generate patterns. Positive instances can also be varied by reversing the action. This stems, of course, from Piaget's definition of operations as internalized actions capable of being reversed. Negative instances are missing at least one defining attribute. For example, the sequence 1, 3, 7, 15 is not arithmetic since the difference between each functional value and its successor is not always the same. In sum, both quantitative variation (varying numbers from instance to instance) and qualitative variation (use of non-instances and reversing actions) are important for operational concept acquisition.

These four instructional factors usually operate together with the possible exception of the embodiment factor. For example, each instance of a concept simultaneously embodies a certain mathematical interpretation, a particular mode of representation, and certain attributes. And in the course of designing instructional/learning strategies for concept acquisition, each factor is subject to instructional manipulation.

There doesn't seem to be any clearly defined end to the exploration phase. Loosely speaking, it is a time for play and organization and can be considered complete when the learner begins to form a rough idea of the concept under consideration. In the case of a figural concept, some success is experienced in correctly classifying instances and in the case of operational concepts some success is experienced in performing the associated operation.

The Assimilation Phase

The assimilation phase spans the acquisition sequence from a rudimentary beginning of a concept to assimilation into an appropriate conceptual structure. It is a rather complicated phase resembling to a considerable extent Piaget's second period of cognitive growth; namely, the formation of representation and operational levels of development. The essence of this phase seems to be properly identifying a concept as an abstraction from objects or from coordinated actions; that is, classifying a concept as figural or operational. This identification will determine the abstraction process and level of internalization (representation or operational).

Figural concepts are abstracted through the process of simple abstraction. In other words, they are assimilated into conceptual structures as objects to be related and acted upon. Let us take for an example the acquisition of the geometric figural concept called trapezoid. From the exploration phase the first instances may have the bases horizontal with the larger base down. There is little noise (irrelevant attributes) in these instances, but instead there is a clear representation of quadrilaterals with two opposite sides being parallel (relevant attributes). This parallel attribute will be abstracted from the selection of quadrilaterals as an internal representation which means that it exists in the mind and can be recalled by thought. The moment in which commonalities of instances are abstracted or internalized varies from learner to learner and depends on the quality and quantity of instances as well as on learner variables, but nevertheless, the abstracting process is similar for each learner--the result of drawing from a number of different situations something which is common to them all. As more and more instances are considered, irrelevant dimensions such as position, relative length of sides, relative measures of angles, and color would be varied so that they are not abstracted as part of the concept. Noninstances, or in this case quadrilaterals with no two opposite sides parallel, are also considered in order to sharpen the boundary of the concept.

Operational concepts are abstractions of coordinated actions that are assimilated at the operation level of thought through the reflective abstraction process. For example, when the sine function has been successfully assimilated through reflective abstraction as an internalized operation, the subject can solve problems of the type, $\sin(x) \approx 0.4$ units as well as problems of the type, $\sin(4^R) = x$ units. However, subjects who solve problems like, $\sin(\frac{\pi}{6}^R) = x$ units by largely memorized data are considered to be at the representation level rather than the operational level. In the abstracting process it is imperative that instances possess the coordinated actions as the commonalities to be abstracted, otherwise the abstraction (if it occurs) will result in a figuration instead of an operation, and the resulting structure will be inadequately internalized. And, as alluded to above, it is important to present instances which clearly illustrate the reverse action of the operation. This will assist the assimilation of the operational concept at the operational level.

For another example of abstracting an operation, consider a function f defined by $f(x) = 2x + 1$ where x represents any real number. If the "understanding" of f results in behaviors such as: $f(2) = 2(2) + 1 = 5$, $f(-3) = 2(-3) + 1 = -5$ (correctly applied, but rotely performed), $f(x + h) = 2x + 1 + h$ or $2x + 2h + 2$, $\frac{f(x)}{x} = f$, and if inverse substitutions such as $f(x) = 6$ as well as responses to questions of the form $f(f(x)) = y$ are only about half correct, then the function f is probably internalized at the representation level rather than at the operational level. The abstraction of f at the operational level must be assisted by a choice of instances which emphasize the coordinated actions and their reverse actions rather than just symbolic representations. Such a choice of instances for $f(x) = 2x + 1$ may emphasize the corresponding coordinated actions as consisting of the following two serially performed actions: first multiply by 2, and then add 1 to the previous product. Furthermore, f is reversible so that the learner should understand that the actions of multiplying by 2 followed by adding 1 can be reversed by first substracting 1 and then dividing by 2.

The assimilation phase is finished when the concept becomes an integral part of the existing conceptual structures. In the case of operations they are assimilated when they have been internalized as operations at which time we say that knowing or understanding has occurred. In the case of figural concepts, assimilation occurs when a representation has been achieved and integrated into an existing structure. Again, we say that knowing or understanding has occurred. Accommodation (probably in an unconscious form) will occur at this time to the extent that existing structures may need to be slightly modified before assimilation can be achieved.

The Formalization Phase

The principal abstracting process of this phase is reflective abstraction considered in the sense of being a mental process of reflection (a reorganization takes place at the level of thought). Skemp (1971) places important emphasis on the ability to reflect on our own mental processes. This ability is generally lacking in learners before they enter the formal operational stage of development. According to Skemp (1971), when learners become able to reflect on their own conceptual structures important further steps can be taken. Structures can be communicated, old structures can be replaced with new ones, and errors can be corrected in existing structures. In general a type of consolidation takes place through strengthening existing bonds and by establishing further bonds with either existing or to be acquired structures. The emphasis is on conscious accommodation and establishing relationships between classes within the conceptual structure. This second order abstraction involves both integration and generalization.

Integration. The process of integration refers to forming relationships between various interpretations of concepts. For example, the concept of function and its associated structures are consolidated when the various interpretations such as mapping, machine, assignment, ordered pairs, and graphs are interrelated. With respect to the sine function for example, suppose the unit circle interpretation has been assimilated into existing structures (phases 1 & 2). The subject can now correctly approximate questions such as $\sin(3^R) \approx x$ and $\sin(x) \approx .1$ by using a concrete unit circle embodiment or an appropriate diagram. Now, suppose a second interpretation of sine, say right triangles, is assimilated (phases 1 & 2 again). When the two internalized interpretations are joined by appropriate connecting relations, the originally assimilated concept of sine has been expanded into a more sophisticated structure which reflects an integration activity of the formalization phase.

Another integrative relationship exists on set-related ideas. This is particularly true with figural concepts. Any two distinct concepts are set-related with respect to their respective instances by either inclusion, partial overlapping, or with no instances in common. For example, the concepts of rectangle and rhombus overlap whereas the concepts of square and circle have no common instances. On the other hand the concepts of square and rectangle are related by set inclusion. Reflective activities which consolidate these examples of set-related concepts within existing structures occur during the formalization phase. Reflective learning processes associated with inclusion-related concepts play a special role and are referred to as generalization processes (Dienes, 1961). Generalization in the contexts of both figural and operational concepts will be discussed below.

Primitive generalization. The process of generalization as applied to figural concepts is exemplified by extending the class of squares to the class of rectangles. Logically speaking, a square is a rectangle since a square has all the attributes for being a rectangle; that is, the opposite pairs of sides of a square are congruent and all its angles are right angles. However, in the generalized rectangle, the opposite pairs of congruent sides are not congruent as they are in a square. Therefore the generalized rectangle is qualitatively different from the square. On the other hand, it would be possible to abstract the concept of rectangles directly, without first abstracting squares, by considering a larger variety of instances during the original abstraction process. Then having acquired rectangles, squares could be defined since its instances were acquired with those of the general rectangle. In short, the process of abstracting square and then generalizing to rectangles is similar to, but distinct from, first abstracting rectangles and then defining squares. We will call the first process of extending one concept to another concept primitive generalization. Attribute variation for primitive generalization appears to be similar to attribute variation involved with the abstraction process. Basically, more and more instances are experienced until the new boundary is clear. Of course, it must be understood that all instances of the original are also instances of the extended concept.

Primitive generalization also is applicable to operational concepts as exemplified by extending the abstraction of exponential multiplication ($2^3 \cdot 2^2 = 2^{3+2}$, $3^4 \cdot 3^1 = 3^{4+1}$, ...) to include all possible cases ($a^m \cdot a^n = a^{m+n}$ where $a \neq 0$, m and n are natural numbers). The first few instances such as $2^3 \cdot 2^2 = 2^{3+2}$, $3^4 \cdot 3^1 = 3^{4+1}$, etc. assist in forming the abstraction of the coordinated action. Then a moment occurs in the abstracting process when the operation is assimilated, but the realization of the extent to which the operation is applicable is in all likelihood unknown (primitive generalization has not occurred). The act of extending the operation to a larger domain of numbers constitutes the primitive generalization process. Although primitive generalization as applied to figural and operational concepts is a similar process, there is one principal distinction. For figural concepts, the original set of instances assisting in forming the abstraction are qualitatively different from the exemplars assisting in the primitive generalization process (in the sense that squares are qualitatively distinct from general rectangles). However, for operational concepts, the original set of instances assisting with the abstraction are by and large qualitatively the same as later instances assisting with primitive generalization (all instances of the form $a^m \cdot a^n = a^{m+n}$ where $a \neq 0$, m and n are natural numbers, are qualitatively the same). In short, primitive generalization for figural concepts requires basically qualitative variation, whereas primitive generalization for operational concepts requires basically quantitative variation.

The process of generalizing to a larger class of instances appears to be cognitively more difficult than the original process of abstracting on a few examples. The final leap to the larger class may require formal operational thought to achieve total primitive generalization since understanding may require the learner to reflect on the form of the transformation rather than performing individual instances of it. The cognitive processes involved with primitive generalizations may be similar to those involved with Piaget's combinatorial system; possession of which partly provides the subject with the ability to reason from hypotheses (all possibilities can be formulated for testing without relying on "concrete" situations to generate hypotheses). From an instructional point of view primitive generalization for operations is assisted primarily through providing a large number of examples.

Mathematical generalization. Finally we will discuss the most sophisticated of the generalization processes, called mathematical generalization, which according to Dienes (1961) can be defined as follows:

> A class B is a mathematical generalization of the class A if B includes as part an isomorphic image of A, in relation to all relevant properties. This means that the classes A and B could consist of quite different elements, as long as there was part of B which was somehow an exact image of A, mirroring the properties of A in all relevant respects (p. 282).

Mathematical generalization is quite distinct from primitive generalization. First, mathematical generalization relates to a combination of figural and operational concepts; in other words mathematical generalization is not applicable to either figural or operational concepts individually. Second, mathematical generalization involves an isomorphic relationship between classes as well as class inclusion. Third, the learner must reflect on the form of an operational concept in order to extend its application to "new" elements of an enlarged class. In order to illustrate some of these distinctive characteristics, let us assume the learner has just primitively generalized the operation denoted by $a^m \cdot a^n = a^{m+n}$, where m and n are natural numbers and a is any non-zero real number. Note that the central object of consideration at this point in the process is an operational rather than a figural concept, although each distinct collection of numbers associated with the operations are figural concepts. The isomorphism from the natural numbers to the positive integers defined by assigning a^{+m} to a^m is necessary from a logical point of view, but in practice it does not seem to have much psychological significance. In other words, the distinction between the operations $a^m \cdot a^n = a^{m+n}$ and $a^{+m} \cdot a^{+n} = a^{(+m)+(+n)}$ as applied to natural numbers or positive integers seems to have a very small psychological role to play in the total mathematical generalization process for this example. From a psychological point of view, the essence of the process begins when the learner reflects on the form of the operation in order to extend its application to "new" types of numbers such as a^0 and a^{-2} which have no sensible interpretation in terms of the original definition, where a^n represents n factors of a and n is a natural number. This reflection on form can be approached in somewhat different ways, but the essential idea is to ignore the original restrictions (to the positive integers at this point of our development) and extend the operation to "new" numbers in such a manner that the extended class of applications includes the original class as a set. Our new applications must be consistent with previous applications so that we obtain an extension in the mathematical sense of the word rather than a totally new logical structure. Psychologically speaking, this can be accomplished by observing what new interpretations must be determined to maintain consistency. For example, $2^3 \cdot 2^0 = 2^{3+0} = 2^3$ by our "illegal" use of the operation, so we conclude that 2^0 must be interpreted as the multiplicative identify, 1, if our desired consistency is to be maintained. Similarly, $2^3 \cdot 2^{-3} = 2^{3+(-3)} = 2^0 = 1$ so that 2^{-3} must be interpreted as the multiplicative inverse, $\frac{1}{2^3}$. The mathematical generalization process is complete when the learner understands that the operation $a^m \cdot a^n = a^{m+n}$ is now applicable for all integers m and n where a is any real number.

Skemp (1971) comments on the sophistication and power of mathematical generalization.

> Sophisticated, because it involves reflecting on the *form* of the method, while temporarily ignoring its content. Powerful, because it makes possible conscious, controlled, and accurate accommodation of one's existing schemas, not only in response to the demands for assimilation of new situations as they are encountered, *but ahead of these demands*, seeking or creating new examples to fit the enlarged concept (p. 61).

This final phase of mathematical generalization involves reflecting on form and the formal properties of conceptual structures. This is one of the most advanced activities of reflective abstraction and requires formal operational thought. Reasoning from a successful generalization is analogous to reasoning from hypotheses. For example, the generalization $a^m \cdot a^n = a^{m+n}$, $a \neq 0$, m and n positive integers, must be understood based on its form rather than depending on "concrete" instances of its use. Citing research by Inhelder and Piaget, Skemp (1971, p. 64) comments that children develop the ability to reflect on content during the ages from about 7 to 11, but could not reason formally -- consider the form of an argument independent of its content -- until adolescence.

Though mathematical generalization related to both figural and operational concepts, it does in fact relate principally to operations. Therefore, whereas figural concepts are abstracted through a two-step extension process: simple abstraction $\Longrightarrow$ primitive generalization; operational concepts may follow a three-step extension process: reflective abstraction $\Longrightarrow$ primitive generalization $\Longrightarrow$ mathematical generalization.

There is no discernible end to this third learning-instructional phase. Further practice and applications will also assist in strengthening and consolidating conceptual structures. In fact, any activity which contributes to further conceptual construction about the original concept could be considered as integrative activity with respect to that concept. In this sense, there is no upper bound to the formalization phase.

Summary

The learning-instructional phases were motiviated by Piaget's three periods of cognitive development. First, initial instances are presented within some planned instructional strategy taking into account the various instructional factors. Second, the concept is abstracted through "concrete" experiences and assimilated into an existing conceptual structure. Third, a reorganization takes place within conceptual structures by integrative processes such as generalization. Simple and reflective abstractions (first sense) constitute cognitive processes dealing with concrete operational thought patterns. Reflective abstraction (second sense) constitutes a cognitive process dealing with formal operational thought patterns.

Implications and Suggested Research

The proposed logical-cognitive view of learning and teaching mathematical concepts was intended to provide orientation and direction for further research rather than to be a logically deduced theory ready for application. Though the learning-instructional phases have immediate application for designing instructional strategies, it is preferred that this point of view be used to generate specific paradigms to serve as guides toward improving the strategy of logical-cognitive type research.

This section is divided into two parts. In the first part specific questions related to various research areas concerning concept acquisition and associated cognitive processes are addressed. In the second part a specific model which appears to have applications for one of the research areas identified in the first part is suggested.

Suggested Problem Areas

Several of the ideas discussed in this part were taken from Beilin's presentation to the initial Models of Learning Mathematics Workshop held at the University of Georgia, May, 1975 (Beilin, 1976).

Research area I. How does information from the world become a figural concept?

Research area II. How do operational concepts come into being either from information from the world or from other concepts? Is reversibility important to operational concept acquisition as suggested in the proposed point of view?

Research area III. How are conceptual structures elaborated, that is, what is the relation of individual concepts to structures that embody groups of concepts? What is the relation of algorithmic learning (learning to perform ordered sequences of explicit procedures such as multiplying whole numbers) to conceptual learning and structural development? For example, does algorithmic learning basically precede conceptual learning or vice versa? Or do they complement each other more or less concurrently?

Research area IV. Does conceptual learning promote retention? How tenuous are assimilated concepts? In other words, how much additional subjection to instances or how much application is necessary to insure permanence of structure?

Research area V. What is the role of definition in conceptual learning and structural development? For example, in passing up a generalized chain of concepts should instances precede definitions? Also, in passing down a "generalized" chain of concepts, should definitions precede instances?

Research area VI. What is the relationship between developmental cognitive structure and mathematical structure? That is, what relationship exists between the knowledge one constructs or abstracts to the processes of obtaining such knowledge? Do the internalized mathematical structures eventually become involved in mathematical reasoning or are the products of thought independent of the processes?

Research area VII. Is mathematical concept learning a function of psychological development? For example, what are the cognitive structures necessary to an understanding of the tenets of trigonometry, or fractions? Are these cognitive structures the same as required for understanding ratios and proportions? Do the processes of generalization (both primitive and mathematical) require formal operational development?

Research area VIII. Are there definitive learning and instructional phases associated with each mathematical concept? Do these alleged phases vary much from concept to concept, and how closely do they parallel Piaget's stages of cognitive development?

Research area IX. What are the roles and relative importance of the instructional factors in concept acquisition (interpretations, embodiments, modalities, and attribute variations)? How much do these factors and their relative importance vary from concept to concept?

Application of a Model

According to Richards (1979), "Models based on theory-prototypes are used in basically two ways. First, as a model for research, it creates testable hypotheses. Second, as a model for learning, it organizes the subject matter according to the theoretical principles." Models provide a framework for research in the sense of simplifying prototypes which are too complex to directly serve as a discipline. But on the other hand models must retain sufficient structure of the prototype to have practical value. In our case, assuming a constructivist's view of the development of knowledge as spontaneous and gradual construction of elementary operational structures, the resulting systems become extremely complex with almost limitless interrelationships involving individual concepts. Furthermore, these interrelationships are constantly subject to change or modification. Therefore models are a virtual necessity.

We wish to propose a model for acquiring chains of generalized concepts related by set inclusion (see research area V). Using Skemp's (1971, p. 25) terminology of "order" as a guide we will call concept A a lower order concept than B (or B is of higher order than A) if all instances of A are also instances of B[7]. For example, the chain composed of square, rectangle, and parallelogram constitutes an ordering of figural concepts from lower to higher orders. The corresponding formulas for finding areas of squares, rectangles and parallelograms constitute a chain of operational concepts. So now questions regarding the role of definition come into focus: can higher order concepts be more effectively acquired from lower order concepts through experiencing additional instances or by relating to a definition? And similarly, can lower order concepts be more effectively acquired from higher order concepts principally through experiencing additional instances or by relating to a definition. According to our point of view, definitions are the principal means of acquiring lower order concepts from higher order concepts since the learner has already experienced and internalized instances of the lower order concept. Therefore, it is only necessary to describe the commonalities of those instances relating to the lower order concept, followed by a few instances for illustrative purposes. On the other hand, higher order concepts are acquired from lower order concepts through the process of primitive generalization which involves experiencing additional instances. Whereas the instances used for acquiring a lower order concept have been internalized a priori, the instances of a higher order concept must still be experienced and internalized before assimilation can occur. In this case definitions are useful in clarifying concept boundaries, but only after the concept has been assimilated.[8]

Research associated with this set theoretical model of generalized chains not only has implications for the prototype in terms of validation, but it also has application to mathematics education curriculum materials such as exist in algebra textbooks. For example, should polynomials be approached through lower ordered concepts such as variables and monomials by using mathematical generalization processes, or should polynomials be assimilated first and then used to acquire the associated lower order concepts through definitions? A similar generalized ordering and related issues exists for relations and functions.

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[7] Skemp used this definition for ordering abstraction levels according to element containment rather than ordering generalization "levels" according to set inclusion. A similar distinction is made by Dienes (1961, p. 293) between the meaning of such phases as "more abstract" and "more general."

[8] Our point of view has implications for some of the conceptual research relating to Henderson's teaching strategies and accompanying sequences of moves. For example, in the context of learning a higher order concept from a lower order one, our prototype implies a "EC strategy" whereas passing down from a lower order chain implies a "CE strategy" (Dossey, 1976, p. 58).

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1. Steffe, L.P. & Hatfield, L.L. A constructivitist model for number and numeration. Paper presented at the Research Workshop: Models of Learning Mathematics, Durham, New Hampshire, October 1977.

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INSTRUCTIONAL AND THEORETICAL IMPLICATIONS OF A MATHEMATICAL MODEL OF COGNITIVE DEVELOPMENT

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Many different theories of cognitive development have been proposed in the past; yet few, if any, mathematical models of intellectual growth have been formulated. Until the recent development of catastrophe theory, existing mathematical methods were not adequate for such a formulation. Two characteristics of human thought contributed to this. First, human cognitive processes are subject to sudden changes, or discontinuities. A familar example of this is a "flash of insight" experienced while solving a problem or learning new material. Second, cognitive growth is considered to be qualitative as well as quantitative. Consequently, any mathematical model not only would have to allow for sudden, rapid changes in thought (i.e., discontinuities) but also be qualitative and generic in nature rather than quantitative. By supplementing equilibrium theory with some results from catastrophe theory, Saari (1976) has created a mathematical model for some of the qualitative, dynamic aspects of cognitive development.

Saari's model seems to provide a useful representation of the dynamics of cognitive development. Its mathematical formulation is intended to clarify and add precision to some aspects of cognitive development that are rather nebulous in existing theories. Unfortunately, its mathematical representation seems to create the illusion that the model is exceedingly complex, requiring extensive knowledge of the mathematics employed, such as catastrophe theory, for its use. Actually, this is not the case. The purpose of this paper is to attempt to remove some of the barriers imposed by the mathematical representation by providing a brief non-mathematical summary and analysis of the model as described in Saari (1976) and extending this model to a characterization of cognitive structures. Then the model will be examined further by looking at some of its instructional implications and its relationship to existing learning theories. Finally, some suggestions for future research will be made.

The Saari Model of Cognitive Development

The Saari model of cognitive development describes the learner as an active interpretor of his environment. Through interaction with the environment the learner constructs a representation or model of the real world, which is the basis for his interpretation of future experiences. The elements of the learner's model world are his cognitive structures; the change in these structures resulting from continued experiences is cognitive development. Hence, the Saari model has two major components: the cognitive structures and the processes by which they change. The latter of these has been described in detail in Saari (1976). A less mathematical summary of this will be presented here followed by a more complete description of the cognitive structures.

Change in Cognitive Structures

Saari's description of the dynamics of the processes essential to cognitive development relies on his mathematical representation of the cognitive structures and is independent of an explicit definition of them. In fact, Saari's representation of structure seems general enough to model any type of mechanism which internalizes and organizes information (e.g. schemas, associative chains, components of long-term memory, etc.). His definition of the cognitive processes does provide the basis for a more explicit characterization of these structures; however, this description will be reserved for a later section. The following examination of the dynamics of cognitive development requires only the mathematical representation of the structures.

Saari assumes that no matter what definition of cognitive structures is chosen, due to human limitations, there can only be a finite number of structures. He then represents the structures as vectors:

> All possible organizational structures are the closed positive orthant of an N dimensional Euclidean space R^N_+. A given organizational structure is represented by a vector $\underline{X}$ from this space, where the k^{th} component of this vector represents the k^{th} organizational structure. $X_k=0$ implies the k^{th} structure is inoperative, while a large value for X_k implies that the k^{th} structure can handle, or permit the interpretation of a large number of messages from the environment. At any given instant of time $\underline{t}$, not all of these potential structures are developed. This is a direct consequence of the fact that the value of N must be so large that all "reasonably" possible organizational structures are represented as a component of vector $\underline{X}$. Consequently, at any time the values of most of the components will be either zero or close to zero (Saari, 1976, p. 123).

The confusions caused by the dual use of "organizational structure" can be reduced by considering each vector $\underline{X}$ as a general type of organizational entity, or structure of structures, with each component, X_k, of a vector representing a specific cognitive structure. Possible representations for the X_k's are "red," "hot," "integer," "place value," as well as more complex ideas.

Each X_k of each $\underline{X}$ has a value denoting its relative utility or strength. A large value for X_k implies it is used to interpret a large number of events in the environment, while a value of 0 means it is inoperative. An environmental event or "message" is a set of stimuli at a particular time; so the same set of stimuli at a different time is a different event. Consequently, the strength of a structure encompasses both the number of times it has been used in a specific situation and the range of different situations to which it has been applied. Note that the actual value of X_k is immaterial; it is its relative size which matters. Larger valued vectors are used more often to interpret the environment than those with smaller values. In short, the magnitude of a component indicates its relative utility as an interpretation of the environment.

Recall that the learner is characterized as an interpreting organism. The learner interprets an event in the environment by assigning it or by matching it to his model world, i.e., his cognitive structures. This matching process is assimilation. Assimilation is thus the link between the real world and the learner's model world.

Assimilation is composed of two potentially conflicting processes, described as follows:

> The first is discriminatory assimilation (As_d) which is the process of examining a message to discover its particular characteristics, to determine how it differs from other messages, etc. The second is generalizing assimilation (As_g) which is the process of searching for agreement between this message and others, of finding a general class to which this message belongs. Clearly these processes change in effectiveness with time due to changes in motivation, etc... All of this can be expressed mathematically as
>
> $$As_d, AS_g: \ E \times R_+ \dashrightarrow R_+^N \quad \text{(Saari, 1976, p. 123).}$$

In other words, the assimilation processes are functions, depending on time (R_+), that map events in the real world (E) into the learner's model of the real world (R_+^N). The structures to which an event is assigned determine the person's perception and understanding of that event. A further assumption is that the effectiveness or sophistication of As_g and As_d change over time. Parameters g and d, respectively, represent some measure of this effectiveness. These parameters can conceivably assume any non-negative real values, with larger values denoting a more motivated, or talented process, while a zero value indicates the process is inactive.

Because each process maps events into the cognitive structures, there is a potential for conflict between the interpretations of a single event by each process:

> If the image of As_g (e,t) agrees completely with that of As_d (e,t), then there are no difficulties, and the message is interpreted as the common image. However, we cannot expect this to always occur, even for the same message

> at a different time. Therefore, in general we can expect that the images of these two processes do not match precisely, that is, they are in at least partial conflict. This state of conflict in the interpretation of a message is our definition of a nontrivial problem.
>
> To resolve this conflict, something in the system has to change. Changes in the organizational structure in response to a nontrivial problem (are) called the accommodation procedure. Namely, it is the process whereby the organizational structures must be modified to fit the message from the environment (Saari, 1976, p. 123-125).

The working of these processes can be crudely demonstrated through the following situation. Standing some distance away, you see a person who looks familiar. Your first reaction is recognition--"That's Jack Jones." This corresponds to As_g matching the whole image of the man to your structures for Jack Jones. However, a second later your reaction is--"No, that's not him is it?" Here, As_d has not matched the details of this event to the structure used by As_g. Hence, conflict in the interpretation which takes the form of uncertainty of recognition.

This conflict between the images of As_d and As_g creates a need for change in the organizational structures. This can be further elaborated by relating As_d and As_g to a particular vector, $\underline{X}$.

By definition, As_d focuses on the details of a particular event, using them to separate this event from all other events, if possible. Hence, accommodation to As_d alone would involve refining the existing cognitive structures. That is, a structure X_k would be divided into two or more new structures, $X_{k_1}, X_{k_2}, \ldots., X_{k_n}$. Consequently, each new structure would be more specialized than the original one. This increased specialization implies that each new structure can interpret a more limited portion of the environment because the new structure has more restrictions. As d increases (that is, as the quality of As_d increases), continued accommodation to only As_d would result in a very large number of specialized structures.

At the other extreme, accommodation to only As_g would produce more general structures. Because As_g emphasizes the similarity between messages, it maps a large number of different events to the same structure. Accommodation to this process alone would then result in structures with only a small number of restrictions that could be used to interpret a large portion of the environment.

Notice that these two assimilation processes balance each other. The generalizing assimilation process coordinates many different events, whereas discriminatory assimilation prevents everything from being related to everything else in one "all encompassing" classification. The accommodation of the organizational structures to As_g and As_d is cognitive development.

The interaction of As_g and As_d in relation to the accommodation process (denoted by Ac) can best be visualized with a three dimensional model (Saari, 1976).[1] To simplify the representation, Figure 1 shows the relationship of As_g, As_d, and Ac with respect to a single organizational structure, X.

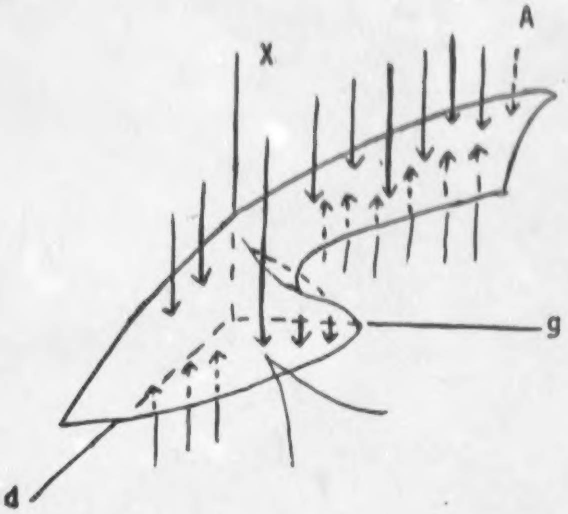

Figure 1. Representation of the equilibrium surface in d-g-X space, N=1.

The surface denotes equilibrium between the assimilation processes. In other words, if a point (d_1, g_1, X_1) is on the equilibrium surface, the interpretation of an event e by As_d and As_g with "qualities" d_1 and g_1, respectively, are in agreement. Hence, e can be matched to X with a value of X_1 without conflict. The event e has been assigned to X by both As_d and As_g. Values of d, g, and X corresponding to a point not on the equilibrium surface imply that As_d and As_g are giving conflicting interpretations of e. X must then change to resolve this conflict. Since adjustment in X can be thought of as an effort to "move" a point not on the surface to a position on the equilibrium surface, the surface is further characterized as an attracting one. This movement toward the surface, denoted by the arrows in Figure 1, is accommodation--the change in the structures to reach equilibrium between As_d and As_g.

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[1] No justification for the modeling of the processes in this manner will be given here. For the mathematical derivation of this representation, see Saari, 1976.

Several characteristics of the equilibrium surface merit special attention. First, the height of the surface (i.e., the value of X) varies greatly, depending on the values of d and g. An increasing value of g with a small d value results in a stronger or more useful organizational structure, denoted by a large component value. In contrast, a small g value with an increasing d produces a structure with a small component value, capable of interpreting only a few events. However, for certain values of d and g, more than one value of X will produce a point on the equilibrium surface. This results from the fold in the surface. This fold is essential in that it represents a discontinuity in the accommodation process. Since accommodation is the process by which the organization structures change, a discontinuity in this process is a rapid change in the structures, causing a sudden change in the value of X. The result of this is a sudden new interpretation of e, which is experienced as a flash of insight or a "bright idea." Notice, however, that this is a representation for a one-structure system. This hardly would model an individual. Therefore, this must be viewed only as a schematic representation.

The most important consequences of this representation are based on the effects of changes in the values of d and g. These effects can be seen more clearly in the projection of the equilibrium surface onto the d-g plane (Figure 2). The cusps, or branches, in Figure 2 are the projections of the boundaries of the cusp folds in Figure 1. The image of any point in the d-g plane can be found by comparing Figures 1 and 2:

> A path moving in the d-g plane will have its image forced toward the surface . . . . The images of the paths from 1-2 and 2-3 can be only on the upper surface. The path from 3 to 4 crosses a branch of the cusp fold. However, this branch corresponds to the fold on the lower surface, so it has no impact upon this path, and the image stays on the upper surface. Notice that path 4-5 crosses the second branch of the cusp, and this branch does correspond to the fold on the upper surface. Therefore, once the path crosses this branch, the adaptation vector field forces the image to the lower surface. Thus, we have the apparent discontinuity in the structure discussed above. (In fact, it is *not* a discontinuity, but rather a relatively rapid change in the value of X.) The analysis of the second path is similar, except notice that the *image of point 4 for the second path differs from that of point 4 for the first path*. Notice that the changes in the values of X are continuous except when a path crosses the second branch of

> a cusp. Thus, the cusps in the d-g plane provide valuable information in that they mark the locations of the discontinuities of the process (Saari, 1976, p. 128).

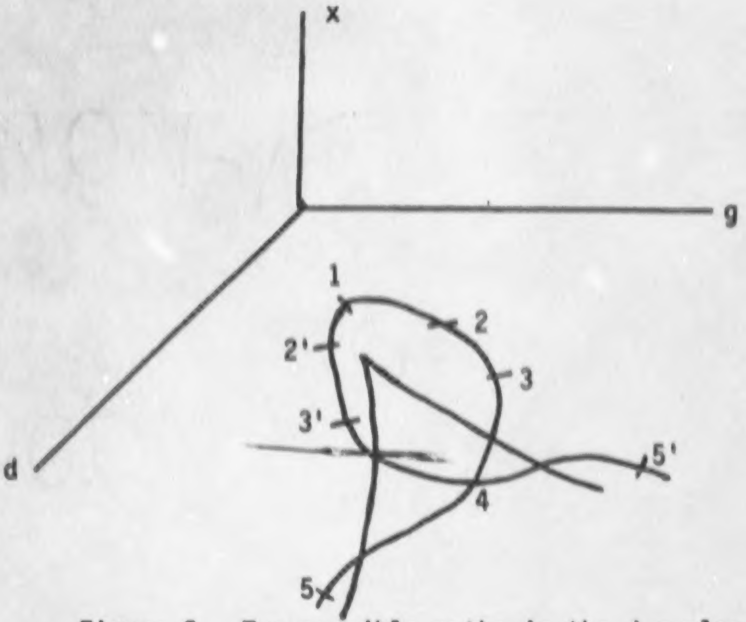

Figure 2. Two possible paths in the d-g plane.

The representation used for a single organizational structure can be generalized to the more realistic situation in which the number of structures is arbitrarily large, but finite. Although the equilibrium surface is too complex to diagram easily, its projection onto the d-g plane is shown roughly in Figure 3. Again the cusps are the projections of the boundaries of folds in the equilibrium surface. It is not clear where the cusps are located, nor how the structures will change. Also, an increasing X value in the one-structure case does not necessarily correspond to increasing component values in the more general case. (A discussion of this last point requires a more detailed description of the cognitive structures, and consequently must be deferred to the next section.) Yet, the important fact in this generalization is that the dynamical description remains the same.

The representation in Figure 3 is intended as a schematic, not as something that can be quantitatively verified. Since an individual has a very large number of cognitive structures, and hence, an N-dimensional equilibrium surface, the location of the various cusps would be impossible to determine. Also, it is highly probable that the locations of the cusps vary from individual to individual. However, what is apparent from this representation is that the sequencing of the use of the processes As_d and As_g makes a critical difference in the development of the cognitive structures. Furthermore, the dramatic "flash of insight" type of learning appears to be the exception, not the rule. Most learning is a gradual

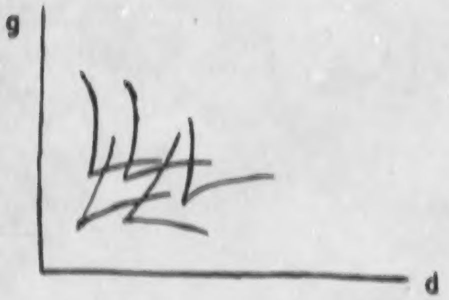

Figure 3. The projection of the equilibrium surface onto the d-g plane when N is arbitrarily finite.

clarification and consolidation of ideas. This more common type of learning corresponds to the gradual change in the structures represented by a movement of a point off the equilibrium surface toward that surface, shown by point A in Figure 1. Thus, the importance of this representation is not that it predicts precisely when a person will experience an "insight" or what it will be, but rather that it allows one to model the sequential use of the processes leading to the development of more general structures, and that it provides a simplified "mental model" with which to visualize changes in the cognitive structures.

The Cognitive Structures[2]

With this description of how the cognitive structures change and of the processes involved, a more detailed characterization of the organizational vectors can be given. Remember, this is a model, not the actual representation of an individual's cognitive structure. Also, it must again be emphasized that "cognitive structure" is being used loosely in that not everything termed a "cognitive structure" will be a structure according to the strict definition of the word (the interrelation of the parts of a whole), though some will be. For present purposes, a structure refers simply to some element in the learner's constructed representation of reality.

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[2] This description stems for a series of conversations with D. Saari.

As defined earlier, a vector $\underline{X}$ is an organizational structure in a broad sense of the word. Each component of $\underline{X}$ is a cognitive structure, or element of "constructed reality." So we have an initial "picture" of $\underline{X}$ as $\underline{X} = (X_1, X_2, \ldots\ldots X_n, X_{n+1}, \ldots\ldots X_r)$ where each component X_k is itself a structure. However, qualitatively different types of elements are represented by various components of $\underline{X}$. Using these different kinds of X_k's, $\underline{X}$ can be divided into a number of segments or levels. Furthermore, this division is hierarchical in that the relationships between the structures at one level form the structures at the next level. That is, the subdivisions in $\underline{X}$ correspond to qualitatively different types of components. This hierarchical nature of the structures can be seen more clearly when a vector is examined in more detail.

The model describes a human being as an active organism, constantly extracting information from the environment through his senses. This sensory input is the most basic and primitive type of message from the environment. A person can receive many sensory messages at once; however, due to human physical limitations the actual number of such identifications can be considered finite. Let N be the number of possible identifications. We can then consider the first N components of $\underline{X}$ (denoted by A) as the sensory components. The particular components associated with an environmental message determine what a person will perceive at a given time. The attributes that an individual notices depend on the availability of the appropriate sensory components. For example, consider a Cuisenaire rod. Each of the attributes of the rod, such as texture, color, shape, etc. are possible sensory identifications. A particular characteristic is noticed when the rod is "matched" to the component associated with that attribute, i.e., the rod is "red" when the person matches it to X_{red}, the component assigned to "redness."

Once these identifications have been made, they can be interrelated by the individual. An identification can be related to itself, or any number of other identifications. However, since there are only N identifications, there are only a finite number, say M, possible relationships between them. Let components X_{N+1} to X_{N+M} (denoted by B) represent all such relationships.

Before giving an example of a B level component, the term "relationship" needs clarification. Relationship refers to logical relationships like implication and set inclusion as well as coordination, in the sense of organizing the incoming information into a meaningful form (e.g., coordinating visual stimuli into a whole image). The B level components are therefore more truly structures than those at the A level. Relationships like "greater than," "younger than," and "same as" are components of the B segment of the vectors.

This definition of the B level of the vectors is actually somewhat debatable. Technically, there are probably several levels that could be inserted between levels A and B, such as levels coordinating isolated sensory inputs into object recognition. However, since the main interest here is in more sophisticated and complex human behaviors, these lower levels of cognitive organization will be left indefinite. No attempt will be made to clarify possible "in between" levels.

The next section of $\underline{X}_i$, C, consists of all the possible relationships between the components in B and A. Note that the C components coordinate relationships. So, for example, the two B level components, $A>B$ (X_j) and $B<A$ (X_K) would then be related through implication at the C level. The resulting C level concept is $A>B \rightarrow B<A$. Note that the C level components better fit definition of "structure" than those of levels A and B.

Similarly, the D level structures are all possible relationships between the C level components. $\underline{X}$ can continue to be dissected in this manner. Looking back, we find that $\underline{X}$ now looks like this:

$$\underline{X} = (\underbrace{X_1 \ldots X_n}_{A}, \underbrace{X_{m+1}, \ldots, X_{n+m}}_{B}, \underbrace{X_{n+m+1}, \ldots, X_q}_{C}, \underbrace{X_{q+1} \ldots X_r}_{D}, \ldots, X_{s+1}, \ldots, X_t)$$

where A = set of sensory identifications
B = set of all possible relationships between A level components.
C = set of all possible relationships between B and A level components,

and so forth. Because there is no known limit to the level of abstraction of human thought, there is no specific number of levels. However, it is safe to assume that there are only a finite number of levels. Also, the number of levels actually used (i.e., containing components X_k such that $X_k \neq 0$) will vary from individual to individual, reflecting individual differences in capability for abstract thought.

Before looking at how this description of the structures relates to the assimilation processes, several important aspects of the structures need to be emphasized. First, structures in each level are qualitatively more sophisticated than those in the previous one. For example, consider $a \rightarrow b \rightarrow c \rightarrow d$, where a,b,c, and d are B level structures related by implication. This string of implications is a level C structure, with a,b,c, and d all being coordinated through implication. It is not $a \rightarrow b$, forming a C level structure then $a \rightarrow b \rightarrow c$, forming a D level structure, etc. Yet $(a \rightarrow b) \rightarrow (c \rightarrow d)$ means that a is related to b, c to d (these are both C level structures) and then that the relationships between the B level structures are being considered, not just the B level structures themselves.

Second, the division of $\underline{X}$ into segments of increasingly more complex structures is not meant to imply that all the lower level structures must become operative before any of those at a higher level may be used. Instead, a type of chain reaction occurs, wherein part or all of a set of components at one level, J, may be coordinated in a number of ways at the next level, J+1; then, in turn, some or all of these may be related in the J+2 level, and so on. The number of "operable" components (i.e., $X_k \neq 0$) and their levels are really indicators of the individual's cognitive development. One would expect an adult to have more structures than a child, due to the adult's greater amount of experiences. Furthermore, as represented by this model, the adult's operable structures are probably spread over more levels, reflecting an increased capability for abstract thought. This is not to say that adults stop creating new lower level structures. Consider a person with expertise in one field who decides to investigate a totally new discipline (e.g., a historian who decides to learn some advanced mathematics). Although this individual most likely operates at a very high level in his own field, in the new area he is probably operating at a relatively low level, building up the lower level structures he is lacking. So, at any point in time, an individual's operable structures can be spread over a variety of levels.

Third, the use of various higher structures depends directly on which A level components are associated with the situation. This comes from the definition of the levels, which specifies that each higher level structure is a relationship between particular structures at lower levels. Hence, every higher level structure is tied to some A level structure, which means use of higher level structures is situation dependent. However, the extent of this dependence varies according to the relative size of the relevant higher and lower level structures. In general, when several structures provide possible interpretations of the same event, the ones with larger values have a greater tendency to be used than those with smaller values. Differences in the relative sizes of higher and lower level components indicate variations in the relative importance of the qualitatively different aspects of the environment they represent. Consequently, variations in the relative magnitudes of higher and lower level structures can result in different interpretations for the same event.

A closer look at the possible combinations of component magnitudes may clarify this. Suppose X_k is a higher level structure that coordinates a number of A level structures. Because X_k is a higher level structure, it is an abstraction; that is, it is a relational concept of some kind. In contrast, the A level structures represent concrete aspects of the environment. In cognitive structures with large values for the A level components and a small value for X_k, the concrete A level interpretations will be used more often than the available higher level structure. X_k would not often be spontaneously applied. Therefore, an event will be interpreted with an emphasis on its details or its concrete aspects, rather than on any higher level relationships. If both the relevant A level components and X_k have large values, then both the details and the relational aspects of the situation are important. In this case, however, the use of X_k is strongly dependent on the A level structures due to their large value. X_k would be used in situations associated with the A level components, but would probably not be applied in appropriate situations whose details considerably differed from those represented by the A level components. Due to their large values, descrepancies between the A level structures and the details of an event would have enough significance to make these components and the related component X_k unsuitable interpretations for the event. Thus, when both the A level components and X_k have large values, the use of X_k is strongly linked to particular instances or types of situations. Finally, suppose the A level structures have very small values in comparison to X_k. The relation represented by X_k dominates the concrete aspects denoted by the A level components. The small values of the A level components mean that the specifics of the environment have little importance compared to the relational aspect represented by X_k. Then X_k could be used to interpret an event whose details differ from the A level components, due to their relative unimportance. In other words, when X_k has a large value compared to the values of the A level components, it is more likely to be used to interpret new situations that differ in their concrete aspects from previous instances in which X_k has been applied.

These differences in situation dependence seem to be demonstrated by the following experiment. When shown a photograph of a boy holding a ruler, some adults were unable to determine the boy's height.[3] Yet when confronted with a real boy and ruler, these same adults had no trouble finding the boy's height. In terms of the model, successful measurement of the live boy means that the cognitive structures essential for measurement were operable and were related to the identifications made. The identifications in the "live" situation were tied to the "measurement" cognitive structures for all the individuals. However, for many of the adults, identifications made in the picture situation were not associated with these measurement structures, as evidenced by their inability to measure the boy in the photograph. These individuals can be characterized as having cognitive structures in which both the lower level structures representing the concrete aspects of the situation and the higher level measurement structures are large. They did apply the measurement structures when the situation was similar to past ones in which they had been used (denoted by the A level structures), but did not apply them in an atypical context. Adults who were successful at both tasks can be characterized as having structures with large higher level components and small lower level ones because the differences in the contextual setting did not hinder the application of the measurement structures.

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[3] This activity was tried with a class of elementary education majors at Northwestern University. More than half of the students were unable to find the height of the boy in the picture, even though he was obviously holding a ruler. It never occurred to these students to iterate the ruler in the picture to determine the boy's height.

This more detailed description of the structural vectors allows a closer look at the accommodation and assimilation processes. As defined earlier, As_g searches for relationships between new and old messages, attempting to combine and interrelate as much information as possible. In short, As_g emphasizes the relational aspects of events. In terms of the structual vectors, relationships between components at a given level are developed through the activation of components at the next highest level. For example, suppose a message e is matched to X_j in segment B of X̲. As_g would attempt to relate X_j to as many other X_i's in B as possible. To do this, the C level structures relating the X_i's to X_i would be used, increasing their value; or, if no such C level structures already existed, new structures at this level would become operable. Hence, As_g coordinates the existing components at one level by using or creating structures at another level. Unrestrained accommodation to As_g alone with g increasing, would result in the coordination of more and more lower level structures by those at higher levels. Then, each of these higher structures would be able to interpret more events in the environment, implying that each component of X̲ would have a larger value.

In contrast, As_d emphasizes the details of events, using them to differentiate events as much as possible. If existing cognitive structures match the details of a situation as determined by As_d, these structures will then be used to interpret the event. Accommodation to As_d in this case increases the values of these structures. If no existing structures fit the details determined by As_d, accommodation then subdivides existing structures into two or more new structures at the same level. In other words, if no existing structures match the details of an event as determined by As_d, accommodation refines existing structures until an adequate fit is achieved. As a result, each new structure would be capable of interpreting fewer events in the environment, decreasing its utility, which would be reflected by a smaller value for each of the new components as compared to the original structure.

Notice that As_d can both increase and decrease components at the same level. (Remember, this discussion refers to the general N-dimensional case.) The main point is that As_d always operates within a level; accommodation to As_d then results in some expansion of that level, either by increasing the value of an existing structure or by developing new structures within that level. Accommodation to As_g involves vertical expansion, creating or reinforcing structures at higher levels.

In general, an organizational structure does not change in isolation. An alteration in the X_i component affects both subsuming and subsumed structures. All the higher level structures relating X_i to other components must change when X_i changes. Similarly, a modification of X_i forces changes in the lower level structures that X_i coordinates. Consider, for example, the concepts "square" and "rectangle," which usually develop (and are taught) as disjoint ideas. The coordination of these concepts as $\{\text{squares}\} \subset \{\text{rectangles}\}$ (or square → rectangle) is going to change the existing structures for square and rectangle as well as activate a higher level structure coordinating them. The rectangle concept structures must certainly change since this coordination means the concept "rectangle" will be less restrictive. The rectangle structure must be modified to allow As_d and As_g to match a square to $X_{\text{rectangle}}$. Also, the rectangle component will increase in value since it can now be applied to more events in the environment.

This description of the cognitive structures and subsequent discussion of the cognitive processes provides a more complete basis than was available in the previous section for interpreting the generalization of Figure 1 to the more realistic N-structure case given in Figure 3. The previous discussion demonstrates the complexity of changes in the cognitive structures in the general case. In the single structure case, movement parallel to the g-axis in the d-g plane produces an increase in the value of the cognitive structure $\underline{X}$; and a move in the d direction causes a decrease in the value of $\underline{X}$. In the general case similar movements do not necessarily result in corresponding increases and decreases in structure magnitudes. The interrelatedness of various changes in the cognitive structures (described above) alone shows that a move in the g direction does not just generate an increase in the value of some structures, nor does a move in the d direction always cause a decrease in structure values. All that can be concluded is that a move in the g direction implies that relational aspects are being emphasized and higher level structures are being stressed. Similarly, movement in the d direction implies that an emphasis is on the details, or on the concrete aspects of a situation. Thus, movement in the g direction can result in a vertical expansion of the cognitive structures, whereas movement in the d direction can generate horizontal expansion of the structures, that is, changes within a level.

Instructional Implications

Mathematics learning has been defined as

> The apprehension of [actual structural relationships between concepts] together with their symbolization and the acquisition of the ability to apply the resulting concepts to real situations occurring in the world (Dienes, 1960, p. 18).

From this definition, understanding a mathematics concept means that the learner should know exemplars of the concept, see its relationship to other concepts, be able to apply the concept in appropriate situations, and yet not apply it in inappropriate situations. In terms of the Saari model, this corresponds to the development of a concept as a higher level structure (i.e., focusing on relationships, not on contextual details) with a value larger than the A level structures it relates (to enable the concept to be applied in a wide variety of situations, rather than being restricted by situational details). On the other hand, this structure must also have some restrictions to prevent it from being applied incorrectly. In other words, mathematical concepts should be developed as higher level structures with values larger than the lower level structures they coordinate with these structures being restricted to prevent their incorrect application.

Introducing a concept as a higher level structure does not necessarily imply it should be presented as abstractly as possible, but rather that it should be treated as an extension of concepts the learner has already acquired. For example, the concept "mathematical group" can be introduced abstractly by giving its definition and the learner may assimilate this as a B level structure--a new entity. However, this concept could also be introduced at a higher level by presenting it as a general description encompassing familiar sets under binary operations, like the non-zero rational numbers under multiplication or the integers under addition. So, definitions should be given in terms of recognized entities from which similar characteristics can be isolated and then emphasized.

Both aspects of the model suggest that to develop a concept as a higher level structure, the learner should start by using As_g. First, consider the description of the structural vectors. Because As_g focuses on the relationships between concepts, an initial use of As_g develops concepts as higher level structures. Introducing ideas at as high a cognitive level as possible also makes the new structure more useful since it will apply to situations already associated with the existing concepts that are being coordinated. For example, consider two B level components B_1 and B_2, that coordinate A level components $A_{11} \ldots A_{1n}$ and $A_{21} \ldots A_{2n}$, respectively. The introduction of a concept L at the B level means the set of A level components related by L must be built up from scratch. Hence, L will have a small value that will be gradually increasing. However, if L can be developed at the C level as a relationship between B_1 and B_2, L will automatically be associated with the A level components related by B_1 and B_2, and hence, will have a larger initial value.

The dynamics of accommodation also suggest advantages to an initial use of As_g by the learner. Again, consider the simple case of a one-structure system (see Figure 4). Recall that creating new higher level structures or more differentiated structures in the general case corresponds to increasing or decreasing values of X, respectively, in the unrealistic one-structure case. Consequently, good concept development in the general case corresponds in the simple case to the development of a structure X_k having as large a value as possible with the limitation that is not applicable to inappropriate situations. Development of a large value for X_k clearly implies an initial move in the g direction in the d-g plane (See Figure 4). But to balance the tendency of As_g to combine everything, the details of the concept should be then explored. That is, a move in the d direction should follow to restrict the applicability of the structure to appropriate situations. Hence, the optimal path for the learner, in general, is one that corresponds to the a-b-c-e path in the schematic diagram in Figure 4.

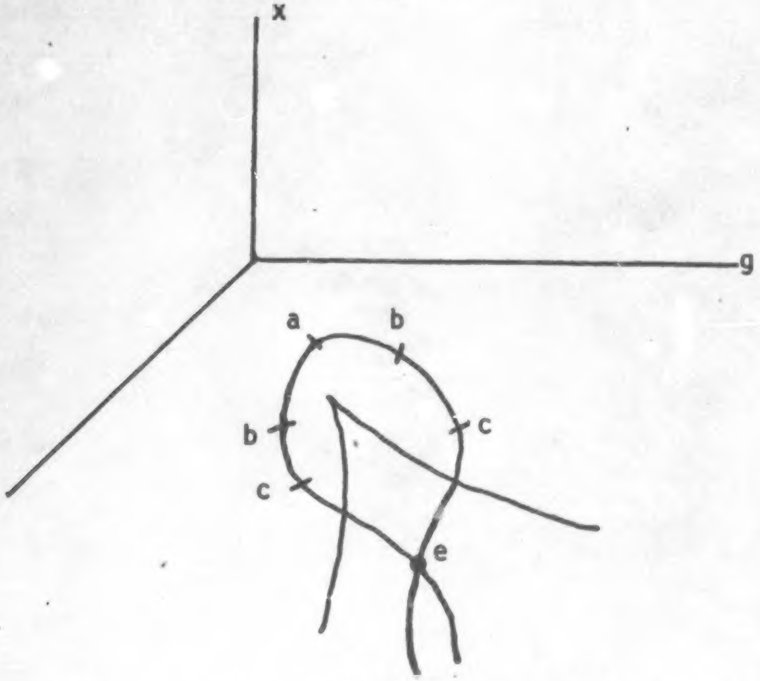

Figure 4. Paths in the d-g plane resulting from different sequences of As_d and As_g.

Note that the image of point e using this path is on the upper level of the equilibrium surface. In the general model, this means higher order implications are being formed, indicating the student has developed a general cognitive structure for the concept. Compare this to path a-b'-c'-e, in which the student first concentrates on the dissimilarities between the new idea and older concepts. Here, the image of point e is on the lower level of the surface, implying the student's structure has a small value, and hence, a very limited understanding of the concept. Even if both paths are identical after point e, the images of the paths remain on different levels of the surface, indicating the importance of the initial movement in the d-g plane. Thus, the model predicts that the student using the first path will develop a more general understanding of the concept in the terms of degree of applicability than will a student using the second path. Therefore, instruction which first encourages the use of As_g and thus first establishes ties between old and new concepts appears to be optimal. All of this assumes, of course, that the model is applicable, and that "correct" structures are being developed.

However, the paths in Figure 4 also indicate that although a sequence beginning with As_g will initially create higher-valued structures, the amount of As_d used at the next step is critical. If at point b or c the learner suddenly stops using As_g and uses As_d exclusively, he could move to points b' or c' which are on the lower level of the surface instead of e, on the higher surface. Hence, in instruction it is important to keep the similarities and relationships "in mind" while differentiating concepts, rather than exclusively focusing on differences.

Another way to view this is that the a-b-c-e path is a representation of the thought pattern encouraged by an instructor who corrects a mistaken "collapse" of concepts (e.g., all rectangles are squares) by showing the student where he has correctly noted similarities and then points out differences (As_d tempered by As_g), as opposed to a teacher who fosters an a-b'-c'-e path by simply telling the student "You're wrong" (massive amount of As_d).

Several other implications from the model have pedagogical significance. Besides the first implication that "better" learning will be achieved when the processes are used as a As_g-As_d sequence, the following implications drawn by Saari (1976) relate directly to instruction:

(2) "Large values of d and/or g tend to lead to more stable structures."

(3) "While the accommodation process is path dependent, it does not depend continuously upon the choice of paths" (p. 131-132).

Each of these will be analyzed in relation to the acquisition of rational number concepts.

First, consider the introduction of the fraction concept. Several different approaches are feasible. One frequently used possibility is to show the student an object, or picture of an object, divided into two equal parts. The child is informed that the object has been divided into "halves," and each part is "one half." Usually, more objects or pictures are presented, with the child being asked to determine those that have been divided into halves. The notation "$\frac{1}{2}$" is introduced almost immediately and "explained." Other fractions are introduced in a similar manner.

An alternate approach is to start with a set of discrete objects, like marbles or beads of several different colors. The concept of fraction can be developed as the number of objects of a particular color out of the total number of objects. In the situation in which two of five marbles were yellow, the student would indicate this and then could record this observation as "2 out of 5" of the marbles are yellow. This type of activity can be done with sets of varying size, with the criteria for division into parts also changing. The observations would continue to be recorded using the "x out of y" notation. After using this notation for some time, the standard fraction notation could be introduced as "convenience," followed by use of this new notation with the discrete set models of fractional parts. The fraction concept can then be extended to the division of wholes into equal parts, etc.

These two approaches encourage the development of qualitatively different cognitive structures. In the first approach, a relatively unfamiliar model, new terminology, and new notation are introduced almost immediately. All of this forces the child to make a number of new identifications. In terms of the model the concept of "half" is being introduced at the A level. Other fractions are introduced in a similar manner, implying that they too are starting as A level concepts. Although the subsequent explanation of the notation "$\frac{x}{y}$" as "x parts out of y parts in all" is an attempt to relate "fraction" to existing whole number concepts, all too often the child fails to grasp this relationship and continues to think of a fraction as unrelated to earlier concepts. (In fact, the model predicts that once ideas have been established as unrelated, considerably more effort is needed to connect them than if they had been interrelated from the outset. This will be discussed as a consequence of implication (2).)

This lack of coordination may stem from a number of factors. For example, "fraction" may be already well established as a separate idea and the experience with interpreting the notation is insufficient for the coordination of "fraction" with whole number concepts. Regardless of the reason, the result is that according to this model, "fraction" is initiated at the A level of the cognitive structures.

In contrast, the second approach develops the concept of fraction as a relationship between two whole numbers; that is "x out of y." No new identifications are initially required. Instead, concepts already firmly established are being related. In other words, as represented by this model, the concept of fraction is being initiated at a higher structural level. The first method began its development of "fraction" at the A level, subsequently requiring the "activation" of B level structures to coordinate the new identifications, and then C level structures to coordinate those at the B level (and hopefully also relate these with existing number concepts). The second approach starts with the development of C level structures coordinating A and B level structures that are already operable. Hence, according to the model, the second approach is superior in that it introduces the fraction concept at a higher cognitive level--at an operational level.

Concept introduction at the A level can have further detrimental effects. In the case of rational number concepts, the A level introduction of other subconcepts such as decimals, fractions as operators, etc., can result in a series of disconnected ideas rather than one all-encompassing concept. Later attempts to coordinate these separate notions often run into failure, as is commonly seen in many "remedial" or "basic" mathematics classes. The reason for this difficulty is more apparent from the second implication from the model and will be examined shortly.

A further consideration is how to help the student already started on a a-b'-c'-e path (see Figure 4) increase his understanding; i.e., how to "move" to the higher surface. Moving to the higher surface from point e calls for the crossing of an additional branch of the cusp. (Remember, a "jump" to a different surface occurs only after crossing the second branch of the cusp.) Crossing this second branch requires an increase in g. Hence, additional experiences relating new concepts to old should be provided. Consequently, students who do not see the connection between whole numbers and fractional parts should be given experiences emphasizing this relationship. However, increasing the g parameter until the second branch of the cusp is crossed is not always the optimal path. A great number of generalizing experiences may be required, making this procedure impractical. "A more efficient path may be one which crosses the second branch of the cusp near the vertex. This requires a decrease in d or a return to the beginning first principles" (Saari, 1976, p. 132). Hence, in attempting to help students with a poor understanding of fractions, this model suggests that "starting fresh" with the discrete model approach advocated earlier may be more successful than attempts to generalize the student's current concept of fraction. This has "common sense" appeal in that struggle with the student's old "hangups" is avoided.

The whole problem of remediation is complicated by the second implication; that is, larger values of d and g produce more stable cognitive structures. As d and g increase, the branches of the cusp may get further and further apart, making the crossing of the critical branch more difficult. Hence, this suggests that the more strongly concepts have been regarded as unrelated, the more difficult it is to coordinate them. Again, consider the concept of rational number. When a student regards each subconcept (fractions, decimals, ratios, etc.) as a separate notion, he has a large value of d and a relatively small one for g. The larger d value implies that his existing cognitive structures for these concepts are relatively stable. Hence, improving his understanding of rational numbers by coordinating these structures would be very difficult.

The stability of structures with large values of d and g further suggests that extra "help" or instruction should be provided at the first indication that a student is failing to coordinate related ideas. A small change in d or g has a large effect near the vertex of the cusp, which is where a concept is more likely to originate. Less effort is required to change suface levels here. Hence, the model suggests that if a child starts to think of fractions as unrelated to whole number concepts, experiences should immediately be provided to emphasize this existing relationship between these ideas. Waiting until the child has further separated these notions (i.e., until d has increased) makes remediation more difficult.

The third implication states that although the accommodation process is dependent upon the path traveled in the d-g plane, the effects of the process are not continuous with respect to the choice of paths. Paths that are close in the d-g plane do not necessarily result in the development of similar cognitive structures. In other words, "individuals with almost identical histories and abilities can reach different interpretations of a message" (Saari, 1976, p. 132). Figure 5, while a schematic representation, illustrates this point.

Paths A and B are almost identical. However, they cross different branches of the cusp near the vertex. The image of B will be on a different surface than the image of A. Hence, extremely similar paths can still result in different levels of comprehension. Notice, this qualitative statement holds no matter where the cusp is located. A typical example of this is a class containing students at various levels of understanding even though all the students were given similar experiences (Saari, 1976, p. 132).[4]

The comparison of the instructional implication drawn above with the results from existing instructional studies could possibly provide empirical evidence with which to examine the validity of the Saari model. Since the acquisition of rational number concepts has been used as an example of the model's instructional implications, actual research into the teaching of this concept was investigated as a logical source of support for these. Unfortunately, the nature of research in this area makes it unsuitable for this purpose for several reasons. First, very little research has actually been done in this area. Second, most of the studies that have been done are comparisons of algorithms for operations with fractions. Unfortunately, investigations of this type provide little insight into the validity of the model. The model predicts that a person's past experience (i.e., path in the d-g plane) has a direct effect on how new situations will be interpreted. This implies that a student's past experience with fractions, such as his grasp of the concept, or work with other algorithms or operations, as well as his understanding of the operation being used, will affect the ease with which a particular algorithm can be learned. Without knowledge of the student's background, no predictions can be made from the model as to which algorithm is best. In addition, the approach used in teaching an algorithm (relating it to an algorithm previously used with whole numbers, for example) may be more important than the actual algorithm used. Few, if any, studies systematically vary the amount of initial relating done. For these three reasons studies concerning rational number concepts cannot provide evidence in agreement or conflict with the model.

Research in concept acquisition would apply most directly to the model. Studies comparing different approaches for initiating the fraction concept, such as those mentioned earlier, would be most beneficial. Several studies along these lines have been done (Galloway, 1975; Muangnapoe, 1975; Payne, et al., 1974; Williams, 1975); however, out of these four

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[4] Since the more realistic representation of the d-g plane is one with numerous cusps as in Figure 3, a number of levels of comprehension are possible.

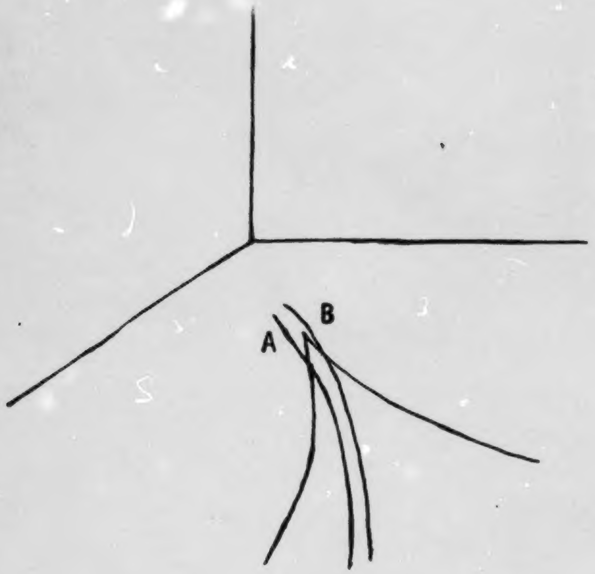

Figure 5. Two almost identical paths which have images on different levels of the surface.

studies only two pilot studies involved an approach emphasizing the relationship between fractional parts and whole numbers similar to that mentioned earlier (Payne, 1976). Yet even these studies began by considering a fraction as part of a region or part of a unit.[5] The essential variable, according to the model, is the relation of the new concept to those previously acquired. Unfortunately, no studies have been found that explicitly investigated this type of variation in the approaches used to teach the fraction concept. Hence, the existing research in the area of rational numbers offers little support for, or detraction from the Saari model.

Related Theories and Research

A comparison of Saari's model to existing theories of learning can be beneficial in several respects. First, such an analysis can add new insights to and understanding of both the established theory and the model. Second, areas of agreement between the theories that are supported by research on the existing theory add validity to the Saari model, just as areas that are contradicted by existing research indicate a need for change in the model. Third, areas of conflict between Saari's model and existing theories will suggest research to resolve these differences. With these points in mind, the Saari model will be briefly discussed in light of the theories of Piaget (1952), Gagné (1970, 1972), and Ausubel (1963, 1978). Because of the qualitative, dynamical nature of Saari's model, only these aspects of existing theories can be compared.

On the surface, Saari's model seems very Piagetian. Both theories portray the individual as an active learner interpreting his environment and describe development in terms of qualitative as well as quantitative growth. The Saari model even sounds Piagetian--assimilation, accommodation, coordination of ideas, etc.; however, it is not a model of Piaget's theory. Although the language used is similar, the processes referred to differ in several important respects. These differences, in accordance with the instructional implication of the model, will be indicated after establishing a fairly large g value in the reader's cognitive processes. Yet, despite these differences, the Saari model describes many characteristics of development mentioned by Piaget and predicts many of his important results.

First, the Saari model predicts behaviors that are important elements of Piagetian theory. As was shown earlier, the Saari model states that use of particular higher level structures is dependent on the situation; that is, the A level identifications determine which higher level structures will be used. This task specificity also has been noted by Piaget (e.g., 1971) in that perceptually different tasks with the same underlying operational structure vary in difficulty. He termed this ability to use a particular operational structure in one situation, but not in another, horizontal declage.

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[5] Although the first pilot seemed to use an approach very similar to the one advocated in this paper, several other problems, such as the presentation of tto much material too quickly, were experienced in this study (Payne, 1976).

In addition, Piaget's different levels of development are reflected in the various segments of the organizational vectors described earlier. Piaget's concrete operations (i.e., internalized actions) are included as B level structures because they can be interpreted as relationships between A level identifications. For example, two rods (each an A level identification) can be related by a size comparison, say "longer than." This comparison, which in Saari's model is a B level structure, is also a Piagetian operation. Naturally, the reverse operation, B is longer than A, is also a B level structure. Since an identification can be related to itself, the identity operation is included at this level. Also, composition is an element of the B segment since there is no restriction on the number of original identifications that can be coordinated by a component at this level. Hence, the set of all relationships between the original identifications contains all the elements of Piaget's concrete operational grouping--an operation, its inverse, the identity, and composition of operations.

The next section of a vector $\underline{X}$, C, consists of the relationships between the components in level B. Since the operation, inverse, identity, and composition are all contained in level B, the components of level C that relate these can be viewed as simple Piagetian groupings. However, more complex structures coordinating several groupings are also components of level C. The coordination of two or more groupings implies the interrelation of the elements of each grouping. In other words, more than four relationships are being coordinated. Since level C includes the relationships between any number of elements of the B segment of $\underline{X}$, the level C components include both simple and complex grouping structures. That is, level C contains the concrete operational structures. It follows that level D would then contain the formal operational structures (operations on operational structures).

The cognitive processes described by Piaget and Saari are another source of similarity. In both models, assimilation is an interpretation of the environment by the individual. Piaget describes four different functional characteristics of assimilation that he calls 1) reproductive or functional assimilation, 2) generalizing assimilation, 3) recognitory assimilation, and 4) reciprocal assimilation (Flavell, 1963, p. 55-58). Functional assimilation refers to the tendency of an assimilatory schema toward repeated application. Generalizing assimilation extends the domain of application of a particular schema to new and different objects. In contrast, schemas are differentiated and refined through recognitory assimilation, which "divides the originally global schema into several new schemas, each with a sharper more discrimination focus in reality" (Flavell, 1963, p. 56). Finally, there is reciprocal assimilation, which is the "uniting relationship between two hitherto separate schemas.... each schema assimilates the other" (Flavell, 1963, p. 57).

There are obvious similarities between Piaget's assimilation functions and As_g and As_d defined earlier. Piaget's recognitory assimilation is certainly very similar to As_d. Both differentiate structures and subdivide existing structures into ones containing more restrictions than the original. However, the functions of generalizing and reciprocal assimilation are both subsumed by As_g. In effect, the Saari model has reduced the number of functional kinds of assimilation one must consider, apparently without losing any of Piaget's qualitative results.[6]

The criticial difference is in the relationship of the assimilation processes to each other. Piaget characterizes the interrelations as follows (these are stated in reference to neonatal reflex schemas):

> In conclusion, assimilation. . . appears in three forms; cumulative repetition, generalization of the activity with incorporation of new objects to it, and finally, motor recognition. But in the last analysis, these three forms are but one: The reflex must be conceived as an organized totality whose nature it is to preserve itself by functioning and consequently to function sooner or later for its own sake (repetition) while incorporating into itself objects propitious to this functioning (generalized assimilation) and discerning situations necessary to certain special modes of its activity (motor recognition) (Piaget, 1952, p. 37-38).
>
> Organization exists within each schema of assimilation since. . . each one constitutes a real whole, bestowing on each element a meaning relating to this totality. But there is above all total organization; that is to say, coordination among the various schemata of assimilation. Now, as we have seen, this coordination is not formed differently from the single schemata, except only that each one comprises the other, in a reciprocal assimilation. . . In short, the conjunction of two cycles or of two schemata is to be conceived as a new totality, self-enclosed (Piaget, 1952, p. 142-143).

Piaget describes a "smooth" interrelationship and functioning of the different aspects of assimilation, whereas Saari allows As_g and As_d to be in conflict.

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[6] In the Saari model, the repetitive nature of assimilation, which Piaget terms "functional assimilation," is a consequence of the mathematical representation of As_d and As_g. Since the processes have not been explicitly defined mathematically in this presentation, this will not be covered here. Personal communication with D. Saari has indicated that a more detailed description of these processes is in progress.

Further contrast between these theories can be found in the definition of accommodation. Recall that the Saari model defines accommodation as the change in the cognitive structures in response to As_g and As_d. In Piagetian theory, accommodation is the "adjustment to the object, . . ., i.e., the organism must accommodate its functioning to the specific contours of the object it is trying assimilate" (Flavell, 1963, p. 46). Accommodation can, however, contribute to the modification of existing cognitive structures in that "the feedback from the new accommodation to the scheme then results in a differentention of the scheme which then begins to function as a new scheme" (Furth, 1969, p. 225). Although in both theories accommodation involves adjustment, Piaget's accommodation is really a mapping from the individual into the environment that can lead to a modification of the cognitive structure, while Saari's is an internal change in the cognitive structures. Piaget describes assimilation and accommodation as reciprocal processes; the former is a function from the environment to the individual's structures, and the latter is a function from the individual's cognitive structures to the environment. Piaget can thus talk about equilibrium between these processes. However, in terms of the Saari model, equilibrium between assimilation and accommodation is meaningless because the first involves functions from the environment into the structures, whereas the second is an internal change; instead equilibrium is the agreement between the potentially conflicting assimilation processes, As_g and As_d.

Thus, it appears that the Saari model contains several of the important elements of Piagetian theory, in spite of the variation in the definition of the cognitive processes. So far, possible losses due to these changes in interpretation are not clear. For example, it seems that Saari's model does not really represent ego-centric behavior. However, the Saari characterization seems to be a simplification in that the critical processes (As_g and As_d) are both expressed as functions with the same domain and range, and the number of assimilation processes has been reduced. The Saari model also gives some "form" to Piaget's rather nebulous cognitive structures and provides clearer prescriptions for effective instruction.

Saari's model also has areas of agreement with the theory of learning and instruction proposed by Gagné (1970). The three principle factors affecting learning described by Gagné--type of learning, conditions internal to the learner, and conditions external to the learner--have parallels with different parts of the Saari model. The first factor, types of learning, is analogous to the various levels of the structural vectors. Gagné's dissection of learning into eight increasingly complex categories is similar to the hierarchical division of the levels of the organizational vectors in Saari's model. In fact, some of the more primitive types of learning described by Gagné (e.g., signal learning, stimulus-response learning, or chaining) could conceivably be the basis for defining levels between segments A and B of the vectors in the Saari model.

Gagné recognizes that the same concept may be acquired as several types of learning. For example, a rule (if A, then B, where A and B are concepts) also could be learned as a verbal association with a lower type of learning. In the Saari model, this corresponds to the possibility of developing concepts at a variety of levels. However, Gagné describes a strictly one-way relationship--lower learning is a basis for more advanced concepts. He omits the reciprocal aspect of learning; that is, that the coordination of lower concepts into more sophisticated ideas also affects the lower structures with respect to those concepts.

The other aspects of Gagné's theory--the importance of internal and external conditions--provide general prescriptions for effective instructon. A few of these suggestions agree with those derived from Saari's model. Both Gagné's theory and the Saari model state that a person's current knowledge should be assessed prior to the presentation of new concepts. This follows from the implications of both theories that new learning should be built upon old, and that each individual has a unique "learning history" (Gagné's internal conditions) which can alter the effectiveness of instruction. Gagné explicitly states that "each learner approaches each new learning task with a different collection of previously learned prerequisite skills. To be effective, therefore, a learning program for each child must take into account what he knows how to do already, and what he doesn't know how to do already" (1972b, p. 75). Thus, Gagné sees individual differences in terms of acquired skills. However, Saari's model characterizes these differences in more general terms; that is, as variations between paths in the d-g plane. Actually, the model's differences are "finer" in that students with different prerequisite skills obviously have different paths; yet, different paths may produce apparently identical prerequisite skills that are really at different levels. For example, addition facts can be learned "meaningfully" or by rote, and as connected to or unrelated to other knowledge about addition.

Gagné's theory has one other point of agreement with the Saari model--the advocation of variety in examples in concept acquisition. He states that "it appears that variety promotes both retention and transfer to other related tasks. . ." (1972a, p. 171). In general, empirical studies of concept and rule learning support this contention (Gagné & Rohwer, 1969). These effects of variety are predicted from the Saari model, because variety in examples promotes the development of large values for the concepts structures. Hence, the structures are more "useful."

At this point, however, the two theories tend to diverge. Gagné advocates diagnostic testing followed by completely individualized instruction based on a predetermined hierarchy of skills. He begins instruction on those concepts at the lowest level of the hierarchy which have not been acquired yet. Note that concepts (i.e., subject matter) form the hierarchical structure, whereas in the Saari model, the learner's interpretation of the subject matter determines the hierarchy.

Furthermore the Saari model makes no specific comment on individualized instruction per se, although it certainly recognizes individual differences. The model stresses the importance of the interrelationship between concepts; however, this is more on the order of an interdependence. As was mentioned earlier, the acquisition of a new concept also increases the comprehension of the older concepts upon which it is based. Hence, the concepts are really integrated, rather than just related hierarchically. Also, since new concepts should be related to those already firmly established, very elementary concepts can be used to develop new ideas. For example, fractions should be derived from whole number concepts and the trigonometric functions should be derived from triangles. Since the model advocates the use of very elementary concepts as bases for new ideas, diagnostic testing may not, and most often will not, be necessary. Therefore, diagnostic testing is not an essential aspect of the model's educational implications.

A different perspective on learning and instruction is provided by Ausubel's theory of meaningful verbal learning (Ausubel, 1963). Several aspects of this theory also are similar to parts of Saari's model. The most obvious of these is the nature of the cognitive structures. Ausubel describes these structures as follows:

> The model of cognitive organization proposed for the learning and retention of meaningful materials assumes the existence of a cognitive structure that is hierarchically organized in terms of highly inclusive conceptual traces under which are subsumed traces of less inclusive subconcepts as well as traces of specific informational data. The major organizational principle, in other words, is that of progressive differentiation of trace systems of a given sphere of knowledge from regions of greater to less inclusiveness, each linked to the next higher step in the hierarchy through a process of subsumption (pp. 24-25).

This structure of increasingly general subsuming structures is strikingly similar to the organization of the cognitive structures described in Saari's model. A close look at both theories suggests many other coinciding aspects.[7] However, one out-growth of Ausubel's theory, the advance organizer, is of special interest in that it provides research results in agreement with instructional implications suggested by Saari's model.

Advance organizers are

> appropriately relevant and inclusive introductory materials that are maximally clear and stable. These organizers are

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[7] In fact, Saari's model may subsume that of Ausubel. Although this cannot be substantiated from the aspects of Saari's model presented here, it is certainly feasible. The major difference is the inclusion of memory and forgetting in Ausubel's model. However, personal communication with D. Saari has indicated that he is incorporating the notion of "memory" into this model.

> normally introduced in advance of the learning material itself and are used to facilitate establishing a meaningful learning set. Advance organizers help the learner to recognize that elements of new learning materials can be meaningfully learned by relating them to specifically relevant aspects of existing cognitive structure. . . . The function of the organizer is to provide ideational scaffolding for the stable incorporation and retention of the more detailed and differentiated material that follows . . . . (Ausubel, Novak & Hanesian, 1978, pp. 170-173).

In terms of Saari's model, the advance organizer would service to initially indicate the appropriate organizational structures for concept acquisition. By suggesting the basis for the development of the new concept, the appropriate conceptual structures coordinating these established ideas become "operable," even though the exact nature of the relationship between these ideas has not yet been specified. Also, briefly relating the new concept to other concepts at the same structural level causes higher level structures which subsume them to become operable. In essence, the advance organizer facilitates the matching of the new instructional material to the cognitive structures by initiating the functioning of the appropriate structures. Hence, the model predicts that the use of advance organizers would facilitate concept acquisition.

Recent studies into the effectiveness of the use of advance organizers in mathematics instruction (Lesh & Johnson, 1976; Lesh, 1976b; Moyer & Johnson, 1978) support this hypothesis. In each of these studies, students who received advance organizers had better concept acquisition (as measured by posttest scores) than students who received post organizers (Lesh & Johnson, 1976; Lesh, 1976a, 1976b) or those who received no organizer at all (Moyer & Johnson, 1978). Since the results of these studies are consistent with the model's predictions, they furnish some empirical support for it.

Thus, Saari's model seems to encompass some of the important qualitative aspects of the learning theories of Piaget, Gagné, and Ausubel. However, there are also some apparent differences, plus many other aspects of the existing theories and the Saari model that have not yet been explored. Although some empirical support for the Saari model has been provided by research pertaining to established theories, additional work is needed in this area.

Future Research

The model of cognitive development proposed by Saari has great potential influence in mathematics education in that it can provide a theoretical basis for the structuring of mathematics instruction. However, a substantial amount of research and "theory building" must first be done. Research essential to the validation and further development of this model of intellectual growth and instruction has two major aspects: 1) research on the theoretical level, and 2) investigations of the instructional implications.

First and foremost, research is needed on the theoretical level. Theories grow and develop through the resolution of conflict between either theory and observation or between two different theories. Hence, for the proposed model to develop, areas of conflict between this model and observations and/or other theories must be identified. This requires a considerable amount of "armchair research" carefully comparing Saari's model to theories from other fields as well as those of cognitive development. How does this model compare with models of perception? Of memory? How can perception and memory be characterized by the model? Research comparing other theories of cognitive development, such as those of Piaget, Ausubel, and Gagne, for example, is necessary. Although a crude beginning of such comparisons was made here, in-depth comparisons are indispensable.

So far only a gross comparison of Saari's model to Piaget's theory has been made. A more thorough and detailed investigation of the commonalities and discrepancies between these theories would be enlightening. So would a characterization of performance on specific Piagetian tasks in terms of the processes described in the model (i.e., As_g and As_d). Such activities could provide answers to a number of important questions such as: Are the interpretations of task performance from Piagetian theory and Saari's model in agreement? If not, what is the nature of the conflict? Does the difference in interpretation of the processes have any effects? An abundance of research questions are evident in this area.

After points of conflict have been determined, empirical studies can be designed to resolve them. Also, the characterization of an observed trait can be hypothesized from the model and then empirically tested. For example, the model implies that exceptional ability in mathematics is directly related to a very strong As_g process; that is, g is very large.[8] Is this accurate? In fact, just a method by which d and g can be measured or compared merits investigation.

Research into the instructional implications of the model also can be done. Again, "armchair research" should preceed empirical studies. A large number of instructional studies already have been done. The implications of these studies in light of the model should be considered before undertaking additional empirical research. Only after the investigation of basic theoretical questions has been done can a comprehensive theory upon which to base the structuring of instruction emerge.

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[8] Krutetskii (1975) also includes this as one of the characteristics of the mathematically gifted.

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A LEARNING MODEL FOR SOME ALGEBRAIC CONCEPTS

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This article attempts to develop a learning model for some algebraic concepts taught at the secondary level. The model is essentially constructivist in the sense that it stresses the importance of the student's need to build meaning for the algebraic forms, the construction of meaning being inherent in the process of understanding.

The first part of this paper addresses itself to the problem of understanding mathematics. It reviews three models of understanding mathematics: Bruner's, Skemp's, and the tetrahedral model. These models identify different modes of understanding mathematics and thus provide a language for its description.

The second part of this article raises the question of the acquisition of understanding and points out some common misconceptions regarding the understanding of algebra.

Finally, under Didactic Reversal, an alternative learning model is presented and applied to the teaching of equations of the first degree in one unknown and to the teaching of the equation of a straight line. This learning model attempts to achieve the accommodation of these algebraic concepts through a process of assimilation, which itself is based on the integration of the four modes of understanding identified earlier.

Models of Understanding Mathematics

One of the objectives on which there seems to be a consensus among mathematics teachers is that they teach for understanding. However, trying to define "understanding of mathematics" is as fruitless a task as trying to define "mathematics." Even a simple question such as "what does it mean to understand equations?" is difficult to answer. In fact, within the same context, different teachers will use different criteria.

As elusive as understanding may be in an abstract sense, in the real world of the teacher in the classroom it becomes concrete. Every experienced teacher has many ways to evaluate it: the kind of questions asked by the students, the answers they give, their attention or lack of it, etc. In fact, no subject could ever be developed by the teacher unless at each step in its exposition there were some indications that the class was following. Of course, this check is not foolproof. Where is the teacher who has never been fooled by the student with the understanding smile, the nodding head?

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The question of understanding has been implicit in the reforms of the last twenty years. These reforms have been primarily concerned with the curriculum. Not only were new topics introduced, but also, many curricular changes took into account various orderings of mathematical concepts in the hope of achieving better understanding. The introduction of "new math" was to provide students with unifying concepts resulting in a more coherent, logical overview of mathematics. Did it?

In a pilot study carried out jointly by Richard Skemp in Great Britain and Stieg Mellin-Olsen (Note 1) in Norway, the relationship between mathematical curriculum and understanding was examined. Both countries had in the early seventies two different mathematics programmes, traditional and modern. Did the choice of syllabus make a difference in the quality of mathematical thinking? The results in both countries were similar and the conclusion was: no, it did not. However, the researchers seemed to find one factor of major importance. "The tentative findings of this pilot study are that what matters most (after pupils' intelligence) is teaching style: that old saying "what matters most is not what you teach, but how you teach it" is true."

Most research in the psychometric tradition has addressed itself to the products of learning, namely the written work produced by the students. These last few years, a new trend in the research in mathematics education has been developing. In various countries such as Great Britain, the United States, Canada, France, and the USSR, some important work is being done on learning models. The emphasis in this work is not on curricular changes but on learning processes. What is being studied is the process of acquisition of a mathematical content rather than the mathematical content itself. These studies are much closer in coming to grips with the problem of understanding than most research on the curriculum. Nevertheless, in examining the total learning process, they still leave the question of understanding in the background.

Some researchers have addressed themselves primarily to the question of understanding. Since this process is too complex to grasp globally, their initial attempts were made through some kind of simplification, through some simple models. Gradually, these models evolve and become more discriminating. There are only a few models dealing directly with the understanding of mathematics. They are essentially qualitative and based on observations of various kinds of behaviour for, as Bruner (1960, p. 29) pointed out, "...how can one know what a student understands save by seeing what he does?" Following is a summary of the Bruner, Skemp, and tetrahedral models.

The Bruner Model

Jerome Bruner does not seem to distinguish between the words "understanding" and "thinking" or between the words "analytic" and "formal." In his book The Process of Education (Bruner, 1960, pp. 55-58), he distinguishes between two modes of thinking: intuitive thinking and analytic thinking. These two modes are complementary and can be contrasted with each other.

Analytic thinking proceeds one step at a time. The steps are explicit and can be expressed adequately. There is an awareness of the operations and the information involved. It may use careful, deductive reasoning and a clear-cut strategy, or it may involve a step-by-step process of induction and experiments.

In contrast, intuitive thinking does not advance in careful well-defined steps. It seems to involve an implicit perception of the total problem. A correct or incorrect answer is obtained with hardly any awareness of the processes used.

Good intuitive thinking seems to depend on prior experience. Familiarity with the structure of a subject makes it possible to leap about, using shortcuts. In mathematics it is also connected with the Eureka phenomenon - the sudden insight, the flash of understanding - and the ability to make good guesses.

Bruner (1973, pp. 85-86) states that "analysis in a well-trained problem solver can be just as activating as intuition, but in many cases it takes a hunch to figure out where the analytic tools should be applied." He also points out that intuition is most successful when it operates in phase with more rigorous and verifiable methods.

Associated with modes of understanding are modes of representation which Bruner (1966, pp. 44-45) perceives as enactive (based on concrete operations), iconic (involving images) and symbolic. He associates the enactive and iconic modes with intuitive thinking and the symbolic mode with analytic thinking. In the usual course of a young child's intellectual development, he moves from enactive through iconic to symbolic representations. If a student has a well-developed symbolic system, it may be possible in teaching to by-pass the first two stages. However, should his symbolic transformations fail to succeed in problem-solving, a student taught only on the symbolic level will be lacking the imagery upon which to fall back (Bruner, 1966, p. 49).

In mathematics education, this interesting link between modes of representation and types of understanding is obviously the result of Bruner's collaboration with Z.P. Dienes (Bruner, 1966, p. 56). Dienes brilliantly demonstrates how mathematical ideas can be given a physical embodiment allowing many children to solve problems through manipulation of the embodiments. The children are then asked to find pictorial representations of the work they have done, and finally in a third stage, are guided in the development of a symbolic representation. Failure to achieve a symbolic representation promptly may be limiting but does not necessarily imply that the intuitive understanding is vague.

"A young student may come to a clear understanding of factoring and prime numbers by visualizing them in terms of the way in which quantities of beans can or cannot be laid out in columns and rows. His understanding is clear but it is limited; it is difficult for him to relate it to other mathematical properties that cannot find an embodiment in the imagery that he has chosen to represent his ideas " (Bruner, 1973, p. 87).

Relationships between ideas to be learned is another aspect of Bruner's work that cannot be ignored. He has written much about the structure of a subject in terms of the organization, the form of a body of knowledge as opposed to discrete atoms of knowledge; the relevance of structure to understanding did not escape him. In his own words: "Grasping the structure of a subject is understanding it in a way that permits many other things to be related to it meaningfully. To learn structure, in short, is to learn how things are related " (Bruner, 1960, p. 7).

The Skemp Model

The British psychologist Richard Skemp has been concerned with understanding of mathematics since the early fifties when, as a mathematics teacher, he was confronted with the problem of children who, though intelligent and hard-working, couldn't do mathematics. In an article published in the British journal, Mathematics Teaching (Skemp, 1976), he describes the evolution of his own thinking:

"It was brought to my attention some years ago by Stieg Mellin-Olsen, of Bergen University, that there are in current use two meanings of this word (understanding). These he distinguishes by calling them "relational understanding" and "instrumental understanding." By the former is meant what I have always meant by understanding, and probably most readers of this article: knowing both what to do and why. Instrumental understanding I would until recently not have regarded as understanding at all. It is what I have, in the past, described as 'rules without reasons,' without realising that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by 'understanding'."

In the pilot study mentioned earlier (Note 1), the two authors justify their terminology by pointing out that many pupils and teachers say they understand something if they can get the right answer to the problems which are set, thus using mathematics in the same sense as learning to use an instrument in order to achieve a desired goal. In contrast to this, relational understanding includes instrumental understanding, but goes beyond it to the underlying mathematical relationship.

Instrumental understanding is indeed a form of understanding since it involves both the recognition of a problem and the use of an appropriate rule. Such learning is part of everyone's experience; everyone can remember learning some mathematics instrumentally, and only at a later stage acquiring some relational understanding about it. Bruner (1966, p. 51) may have somewhat overstated this by writing "most learning starts off rather piecemeal without the integration of component acts or elements."

It can be claimed that even mathematicians use some mathematics instrumentally in their work, that is they apply remembered rules. However, this interpretation must be considered distinct from the instrumental understanding which may be involved in an initial learning experience; after all, mathematicians have the ability to derive the rules they use.

To what extent is the distinction between types of understanding important? In any educational situation, essential components are the child, the teacher, and the educational material (textbooks, worksheets, games, etc.). Skemp (1976) points out that among these three components there is the possibility of a mismatch. If the kind of understanding aimed at by one of these components does not correspond to the one aimed at by the other two, there is a mismatch in the objectives.

An example illustrating the difference between the two modes of understanding will be useful. "It is certainly easier for pupils to learn that 'area of a triangle = 1/2 base x height' than to learn why this is so. But they then have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all of these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not want to have to derive them afresh everytime " (Skemp, 1976). Skemp examines the advantages of each approach:

Using an instrumental approach, it is usually easier to learn a rule, such as the product of two negative integers, than to find a relational justification. An instrumental approach is more efficient, more reliable and yields immediate answers.

On the other hand, a relational understanding is more flexible; it can handle new problems. Mathematics understood relationally is easier to remember, for it possesses an inherent structure and organization. It can be far more satisfying intellectually and may lead to further growth and discovery.

The Tetrahedral Model

In an article titled "Understanding School Mathematics," Byers and Herscovics (1977) introduce what might be considered a second generation model, inasmuch as it builds on the previous ones. Bruner and Skemp each identify two modes of understanding which describe different but complementary aspects of comprehension. The tetrahedral model is not simply their union; the concept of analytic understanding is now incorporated in the broader concept of formal understanding.

It was initially suggested by Byers that a good model of understanding had to take into account the unique nature of mathematics as a formal science; it had to distinguish clearly between mathematical content and mathematical form. To some extent Bruner's and Dienes' work on modes of representation reflects this distinction, but their approach concerns primarily the development of the young child, not the different mathematical modes of representation.

Mathematical form involves both the question of representation, the notation and symbolism used to represent mathematical concepts, and the sequence, the order in which these are linked together into chains of logical reasoning. For instance, CCXXXIV ÷ XVIII and 234 ÷ 18 are distinct mathematical representations of the same problem. A numerical table, a graph, or an equation can represent the same function. Quite likely, the development of representation follows some natural law involving the economy and power of the notation, economy in the sense of carrying an optimal amount of information, power in the sense of generating new ideas.

The tetrahedral model is based on the following four definitions:

Instrumental understanding is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.

Relational understanding is the ability to deduce specific rules or procedures from more general mathematical relationships.

Intuitive understanding is the ability to solve a problem without prior analysis of the problem.

Formal understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning.

It is merely for the purpose of visualization that the creators of this model suggest that understanding can be viewed as a tetrahedron whose four vertices correspond to the four modes of understanding. The state of understanding of a concept at any given time can be considered as a point within the tetrahedron. This point however is not static, for any additional knowledge not only increases the comprehension space (the size of the tetrahedron), but also affects the modes of understanding. These can be considered as four different functions varying with time. This model reflects both the dynamic nature of understanding and the fact that it usually involves more than one mode.

Another aspect which cannot be ignored is the interactions among the four modes of understanding. They either reinforce or hinder each other. For instance, an intuitive approach may help achieve relational and formal understanding. Connecting a notation with the relevant mathematical ideas may help develop a better relational understanding of these ideas. On the other hand, a formalistic approach may have negative effects and lead to pushing around meaningless symbols or parroting senseless definitions.

The interactions among the various modes of understanding have pedagogical importance. For example, a clearer view of what kind of understanding is desired in the development of a topic may help the teacher in tailoring his presentation to his objective. He may help develop an intuitive understanding for a topic by encouraging the use of imagery and pattern recognition, while other modes of understanding of the same topic would require different approaches. The transitions from intuitive to formal understanding and from instrumental to relational are difficult and require time. These transitions are seldom achieved through textbooks or programmed instruction. The teacher is still irreplaceable.

Byers and Herscovics (1977) stress that effective learning cannot be based on any one type of understanding; rather, all four are essential. The problem is that of integrating the four modes. "Regardless of teacher preference and social pressure, it would be dangerous to set any point in the tetrahedral space as the goal in mathematics instruction. Optimal learning appears to require a spiral approach in which the different types of understanding are used consecutively and repeatedly at ever greater depth " (Byers and Herscovics, 1977, p. 27).

Pedagogical Implications

The understanding of mathematics has remained an objective of the teaching profession. Because it has been so difficult to define understanding, it has often been expressed in the form of vague generalities. This inability to define understanding clearly has made it difficult to counter the criticisms of those evaluators using exclusively as their criteria the so-called products of learning, namely the right answers. The models bring greater precision to the notion of understanding and provide a language and a frame of reference for discussion and reflection.

In mathematics, once something has been understood, it has a tendency to seem trivially simple. This makes it extremely difficult for teachers, who do understand mathematical concepts, to evaluate properly the obstacles confronting the learner. This problem is compounded by the fact that, at least at the high school level, mathematics teachers have mathematical abilities far exceeding those of the general population. Thus, teachers have had a relatively easy task in achieving their formal level of understanding. It is probably because, once acquired, this mode is so economical and powerful that it is so hard to shift to other modes. This may also explain why so many topics are taught formally. The models may create a greater awareness of the need to communicate at a level accessible to the learner. The four modes of understanding can become distinct objectives and be reflected in various teaching styles.

Acquisition of Understanding

Although the models describe different modes of understanding, they do not explain how it occurs. Thus the problem of learning mathematics must be viewed in the broader context of cognitive growth. In Piaget's theory, cognitive growth occurs through the interaction between the individual and his environment, through the process of equilibration which involves the integration of new knowledge and new experiences with already possessed knowledge. Piaget calls assimilation the act of fitting the new material into the individual's existing schemas (mental structures). If assimilation is not possible, then the need to integrate leads to a much more difficult problem, that of changing one's schemas, that of building new ones - namely accommodation. How this is achieved in general is very poorly known.

In a recent paper on learning models, Leslie Steffe (Note 2) has framed this question in more practical terms. His learning model starts with an exploratory phase in which the mathematics involved is new to the child. If the new material can be connected to a simpler idea existing in the child's cognition, then this simpler idea can be modified and expanded to integrate the new one in a meaningful way. On the other hand, if the new material cannot be connected to a simpler idea in the child's cognition, the problem which arises is that of contructing meaning for the concept or operation involved, as well as for its form. This is strikingly illustrated in Herbert Ginsburg's (1977) book describing how children learn arithmetic. Even before entering school, children first memorize number words and eventually their sequence. They then learn to apply these by counting small numbers of objects. Over a period of time, children learn to "see" small numbers directly. Thus, they have constructed a meaning for what was initially a meaningless sequence of sounds. However, even then, their concept of number is not the same as that of an adult's. Piagetian work on the lack of conservation of number is well known.

One can introduce a new subject by connecting it to a simpler concept or to an equivalent concept known to the student. To deal with a new subject by expanding a simpler one involves concept formation. For instance, the initial meaning associated with multiplication is that of repeated addition. On the other hand, to connect a new concept with an equivalent one can be viewed as a problem of representation. For example, straight lines in the plane and linear equations in two variables can be considered as two representations of the same concept.

In order to establish these connections, one often has the choice of two possible approaches. Starting from the new topic, one can transform it to reach the student's cognition, or, starting from the student's cognition one can transform it to reach the new topic. If the new material is not yet understood, the transformations performed on it run the risk of concealing its significance and create a problem of an accommodation which may be too large for the student to accomplish. Alternately, transformations performed on the student's cognition in order to reach the new topic allow him to construct meaning and thus represent a process of assimilation. This process of assimilation can lead to the desired change in the learner's schemas for as indicated by Piaget "effective assimilation has as its counterpart a more or less effective accommodation" (Piaget & Inhelder, 1967, p. 455).

These theoretical considerations have been reflected in some of the research aimed at younger children and have resulted in a number of pedagogical innovations at the elementary level. The operational nature of the learning process often requires the manipulation of concrete materials as a starting point and many schools now have access to a wide variety of teaching aids. Better yet, many teachers and their pupils learn to construct them. Creative play and guided exploration have become important teaching methods; these methods encourage the development of the child's intuition. On the other hand, at the secondary level, most changes have been of a curricular nature. The psychological and pedagogical problems have received relatively little attention. The transition from elementary school to secondary school is quite dramatic. No more exploratory activities, no more reliance on concrete materials, subjects are taught by specialists.

Some Misconceptions

While arithmetic and measurement form the core of the elementary programmes, those at the secondary level are characterized by algebra and geometry. Of these two subjects, it is the teaching of geometry that has seen some innovative and fresh approaches; the teaching of algebra has seen little change other than the introduction of set and functional notation. Yet it is algebra which constitutes a major stumbling block in the mathematical development of a great many students. It is not too difficult to identify the main reason for this: the students are confronted with a mathematical form totally new to them. Bezuszka (Note 3) has a telling anecdote of a teacher saying to his class "Let x by any number" and a student responding with "Sir, is that the same as saying, 'let 3 be any letter'?"

Attempts are made to make algebra relevant by introducing simple word problems. "If a pen costs 11 cents more than a pencil, and together they cost 45 cents, how much does each one cost?" "If a pound of coffee costs $4, and tea costs $3 a pound, how much can be bought with $12?" No doubt, this approach is successful with some pupils. However, a large number of students find it difficult to translate these problems into a language they don't understand. It may be as difficult for them to come up with $2x + 11 = 45$, $4x + 3y = 12$, as it would be for teachers to translate the problems into ancient Egyptian, using hieroglyphs.

One of the reasons teachers underestimate the learning difficulties involved with these problems is that they falsely assume that elementary school children have a good grasp of equality and the concept of unknown.

Very early, at the elementary level, children are exposed to the equality sign, as in $3 + 5 = 8$. However, that their notion of equality is different from an adult's is reflected in the way they read this: "3 and 5 *make* 8." Ginsburg (1977) points out that children's understanding of *symbols* refers to actions, to calculational operations. Children can also solve $2 + \boxed{} = 7$ or $3 \times \boxed{} = 6$. This fact is mistakenly interpreted as indicating some *understanding* of the concept of an unknown. However, since many students cannot go further in algebra, this must be construed as being simply a form of mental arithmetic, not algebra.

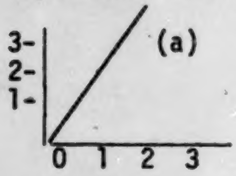
Again, the fact that children learn to identify ordered pairs of numbers with the coordinates of points in the Cartesian plane leads many teachers to believe that the transition from a linear equation to its graph is but an easy conceptual extension. All that is needed is a few ordered pairs obtained from the equation, plotting the corresponding points and, lo and behold, see how they're all on a straight line. Yet, if the linear equation is totally meaningless, then so is the derivation of the ordered pairs. The reason for which these ordered pairs no longer correspond to random points in the plane may be completely lost on the student.

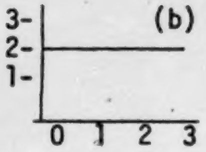
By and large, algebra is taught formally; students are expected to gain understanding by manipulating algebraic forms. Those who do not succeed abandon mathematics as soon as the school system will allow it, only to be confronted with the same problems when they enroll in the social science or the commerce programmes at the university.

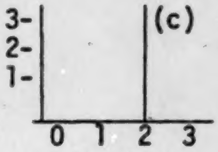
What about those students who have managed to acquire some instrumental understanding? Recent research has brought into question their real level of comprehension.

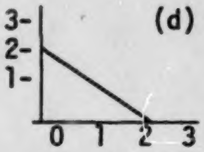
In a paper on mathematical language acquisition, Sigrid Wagner (Note 4) points out that the use of variables distinguishes mathematical language from ordinary verbal language, and she suggests that the concept of mathematical variable is not a natural one. Every once in a while, someone comes up with a very simple question which proves to be a real pedagogical eye opener. Wagner has come up with such a question. She asked students whether $7N + 22 = 109$ and $7W + 22 = 109$ yield the same solution or not. The answers varied widely: "the solution of the second equation is greater than the first one because w comes later than n in the alphabet;" "can't tell until both equations are solved;" "of course, the solution is the same" (Wagner, note 5).

Do students understand what is meant by an equation and its graph? Daphne Kerslake (1977) tested 600 pupils aged 12 to 15 in four British secondary schools. She asked them: "which of the following graphs represents the line $y = 2x$?"









Percentage of correct responses:

	Year 2	Year 3	Year 4
Boys	13.8	35.4	36.0
Girls	21.6	19.4	36.8

Kerslake (1977) also has results indicating that most students have no understanding of continuity; they do not view a straight line as an infinite set of points. Thus it is apparent that algebra is taught formally but no formal understanding occurs.

The question which arises is whether such lack of understanding is a reflection of the student's mathematical abilities, or perhaps is a result of the way they are taught? It is the contention of this article that a formal approach to the teaching of algebra, i.e. the introduction of meaningless form, leads to an instrumental understanding and can reach but a small number of students. Alternative approaches should be sought and the last section of this paper is one such attempt. It introduces a learning model for the teaching of some algebraic concepts: didactic reversal.

Didactic Reversal

The algebra taught in most secondary schools can be divided into two major levels. At first, the student is introduced to the concept of an equation in one unknown and at a later stage, he is presented with algebraic equations in two variables.

The teaching of equations starts with an equation, say $2x + 7 = 13$, which through some manipulations yields the answer $x = 3$. The student then verifies his answer by substituting it in the equation and obtains the arithmetic identity $2 \cdot 3 + 7 = 13$. He has gone from an algebraic form to an arithmetic form, i.e., from a new form to a simpler old form he knew.

When learning linear equations, say $y = 2x + 3$, he finds a few ordered pairs satisfying the equation, plots the corresponding points, and obtains the corresponding straight line. He has gone from an algebraic form to a geometric form, or from a new form to an equivalent form which he already knew.

In such situations, the pupil finds himself facing a prefabricated algebra whose notation hides mathematical concepts which he must uncover. Constantly going from algebra to other modes of representation, it is difficult for him to conceive of the possible equivalence between these other representations and the algebraic representations.

If the very concept of an algebraic equation, in one unknown or in two variables, is totally new and meaningless to the student, then the problem he faces is one of accommodation.

Didactic reversal is essentially a learning model which views algebra as another representation of arithmetic and geometric ideas. It attempts to achieve accommodation through assimilation by taking as a starting point those mathematical representations which already exist in the student's cognition. At first, in small manageable steps, the student is guided in the construction of the algebraic form by using his knowledge of arithmetic and geometry. This allows him to confront mathematical concepts and operations before these are frozen in a symbolism difficult to understand. Moreover, such an approach marks algebra as being a representation of arithmetic and geometric ideas. By gradually constructing meaning for this new material, the student reaches a certain level of formal understanding.

It is only when the algebraic form has acquired meaning that reversal is encouraged. That is, starting now from meaningful algebraic forms, the student is asked to find the arithmetic or geometric representations.

Following such a learning scheme could lead the student to conceive explicitly of the equivalence of different representations. Any arithmetic identity can be transformed into a first degree equation in one unknown and, any such equation can be transformed into an arithmetic identity upon substituting its solution for the unknown (excepting equations such as $2 + x = x + 2$ reflecting the axioms, or equations such as $x + 2 = x + 3$, which have no solution). Similarly, every straight line in the plane can be represented by a linear equation in two variables and conversely, any such equation can be represented by its graph.

Such equivalences form important links between the different mathematical subjects taught in the schools. This going back and forth between different representations gives mathematics a dynamic and operational flavor which is lost in the definition-and-rules approach.

Applying didactic reversal is no easy task. For each new topic the teacher has to identify those simpler or equivalent presentations which are known to the student. He must then discover how to transform these in order to lead the student into constructing meaning for the new subject.

The following examples illustrate how didactic reversal can be applied to the teaching of equations.

First degree equations in one unknown

In applying didactic reversal to the teaching of first degree equations in one unknown one must seek an equivalent or simpler concept existing in the student's cognition. The concept of an arithmetic identity is familiar to him but exists at a fairly primitive level, that of an operation on one side of the equality and the result on the other ($2 \cdot 3 = 6$). In order to prepare the construction of meaning for a non-trivial equation, it is necessary to expand the pupil's meaning for the equality symbol. This is achieved by asking him to build a variety of arithmetic identities involving multiple operations ($3 \cdot 4 + 5 = 20 - 3$). Since to many students the word "equation" refers to an arithmetic identity, it is useful to introduce the words "arithmetic identity," which are not difficult to justify since both sides of the equality sign yield the same number.

The concept of an equation can now be introduced in small steps leading up to the new mathematical form. An arithmetic identity is chosen and one of its numbers is covered up with a finger. Thus an equation is defined as an "arithmetic identity with a hidden number." Defining equations in such a simple way allows the student to grasp the concept prior to being confronted with the new form. He can now seek more practical representations and replace the hiding finger with the familiar empty box. The empty box is an intermediate step and is then replaced by any letter chosen by the student. The concept and terminology of an unknown are now more readily assimilable. Since the new symbols have acquired some meaning, the student has now achieved some level of formal understanding.

It is useful in this case to follow Mick and Brazier's (Note 6) classification of mathematical concepts into figural concepts (mathematical forms) and operational concepts (operations performed on mathematical forms). Thus one can distinguish between an equation and the operations required to solve it. It is not sufficient to have established meaning for the figural concept, the algebraic operations also need justification.

In a thesis now in preparation, Kieran (Note 7) points out that the operations used in solving equations can be linked to some operations performed on arithmetic identities. Given any arithmetic identity, one can use it to construct new identities by performing the same operation with the same number on both sides. For instance, $3 \cdot 4 + 5 = 20 - 3$ can lead to $3 \cdot 4 + 5 + 10 = 20 - 3 + 10$ or $3 \cdot 4 + 5 - 7 = 20 - 3 - 7$. This kind of arithmetic intuition provides a justification for the techniques used in the solution of equations since these are now conceived as arithmetic identities with a hidden number.

For small values in N, students have no difficulty in solving simple equations such as $x + a = b$, $x - a = b$, $ax = b$, $x \div a = b$ although $a - x = b$, and $a \div x = b$ seem somewhat more difficult. In fact, their "guessing" of the answer is nothing but an extension of mental arithmetic and they often do not see much point in the formalization (i.e. the writing out of the required steps) requested by their teacher. It is when the numbers used are large enough or when multiple operations appear in the equation or when terms in the unknown appear on both sides ($ax + b = cx + d$) that mental arithmetic usually ceases to work and algebraic methods come into their own. The gathering of like terms reduces the equation to an easy-to-solve type. Since equations are identified with arithmetic identities, the rules developed earlier can now be used for the gathering of like terms, giving the student a relational understanding for his operations. By introducing equations restricted to the natural numbers, one avoids the arithmetic difficulties some students face with negative integers and rationals. Of course, the majority will be able to move on to solving equations in higher number systems. In fact, some students are so proficient in mental arithmetic that it is only when solving equations in Z or Q that they can be convinced to learn algebraic methods.

Students can readily accept that to solve an equation means to recover the arithmetic identity and that this can be achieved by gathering like terms. However, as with all other problems whose solution involves multiple steps, the ability to justify each step does not imply that the student has acquired an overview of the various possible strategies needed in the solution process. Some students can develop these on their own, but others need to be taught. However, when these strategies are taught in the context of didactic reversal they become more than meaningless manipulations.

Linear equations in two variables

Applying didactic reversal to the teaching of linear equations in two variables means starting from geometric concepts known to the students.

High school students have relatively little difficulty in relating points in the Cartesian plane with their coordinates in N, Z, or Q. They may have difficulty with the notion of a functional variable (Wagner, Note 4) but not with the concept of a variable point in the plane as distinguished from a fixed point in the plane. They realize without difficulty that an ordered pair of numbers fixes a point in the plane and thus accept the use of letters for the coordinates of a variable point. The letters x and y are the logical choice reflecting the respective axes.

In order to construct the equation of a straight line, the student must realize that it is merely a rule describing a relationship between the coordinates of any point on and only on the given line. This creates a pedagogical problem for, as Kerslake's (1977) research shows, the majority of students may not think of a straight line as an infinite set of points. However, Piaget and Inhelder have shown (1967, p. 125-154) that from about the ages of 11 or 12, children can fathom the possibility of drawing an infinite number of points on a straight line. Such an operational conception can be used to derive intuitively the equation of a straight line.

Students are first asked to "cover" with points a given straight line through the origin and to find the coordinates of some of these points. They need but few of them, say (1,2), (1.5,3), (2,4), (3,6), to recognize a pattern and verbalize a rule connecting the coordinates: "the second one is double the first one" or "the first one is half the second one." Using now x and y as the coordinates of any point on the line leads to the formalisation $y = 2x$. It is useful at this stage to ask the students if the coordinates of points in the plane which are not on the line will satisfy this rule. Some students are also quite surprised to discover that ordered pairs derived from their equation will yield points on the line. Even when the line $y = 2x$ is translated, its new equation $y = 2x + b$ can be "guessed" by an average student who has been shown how to break up this problem into two consecutive steps. Thus, if recognition of number patterns can be interpreted as intuitive understanding, then this intuitive understanding leads to a level of formal understanding when the student expresses the relationship between the coordinates in the form of an equation.

This preliminary work helps the student construct the equation of a straight line but does not tell him why its form is linear. Such a level of relational understanding can be achieved through the introduction of the concept of slope and its invariance.

Following a suggestion by Petsu (1977), the concept of slope can be introduced at an intuitive level by using the notion of a staircase. Its measure is easily defined in terms of the ratio of the riser to the tread. By attaching such a staircase to a straight line in the Cartesian plane, the slope of the straight line can easily be measured. Very soon, the staircase can be discarded since one of its steps is sufficient to measure the slope.

If students have not yet had geometry, the proportionality of similar triangles cannot be used to prove the invariance of the slope. However, by varying the position and the size of the step along the line, students find it easy to accept this invariance. Some challenging, but seldom raised, problems can now be asked. Will a line whose slope is invariant necessarily be a straight line? Can one draw a straight line whose slope is 2, 3/2, - 2/3...? Is the slope of a straight line sufficient to determine it uniquely? What else is needed?

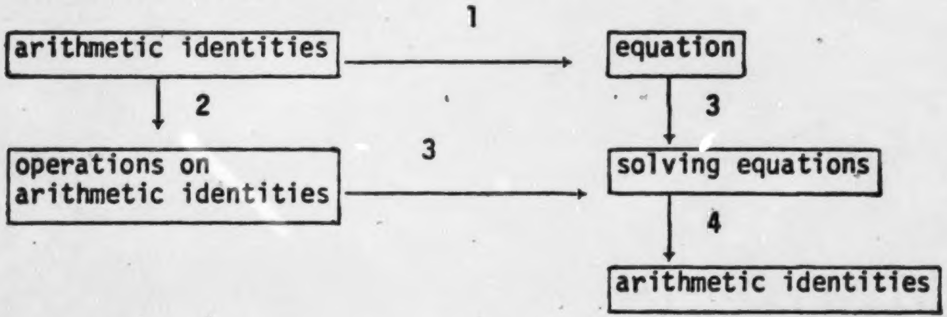
By now selecting a fixed point (x_1,y_1) and a different but variable point (x,y) on a straight line with slope m, the invariance of the slope yields $\frac{y - y_1}{x - x_1} = m$. The point-slope formula can then be transformed to the slope-intercept formula $(y = mx + b)$ and to the general linear equation $ax + by = c$.

Horizontal and vertical lines are better treated as a special case. Some students find it difficult to accept that the slope of a vertical line cannot be defined. This is due to a lack of distinction between inclination and slope. It is easier to start with horizontal lines and derive their equation $y = k$, for then the transfer to the equation of vertical lines becomes acceptable.

At this stage the new algebraic form has acquired meaning and has become justifiable. The time is now ripe for reversing the process. Given a linear equation, the student can now recognize its form and is aware that it has infinitely many solutions whose graph is a straight line.

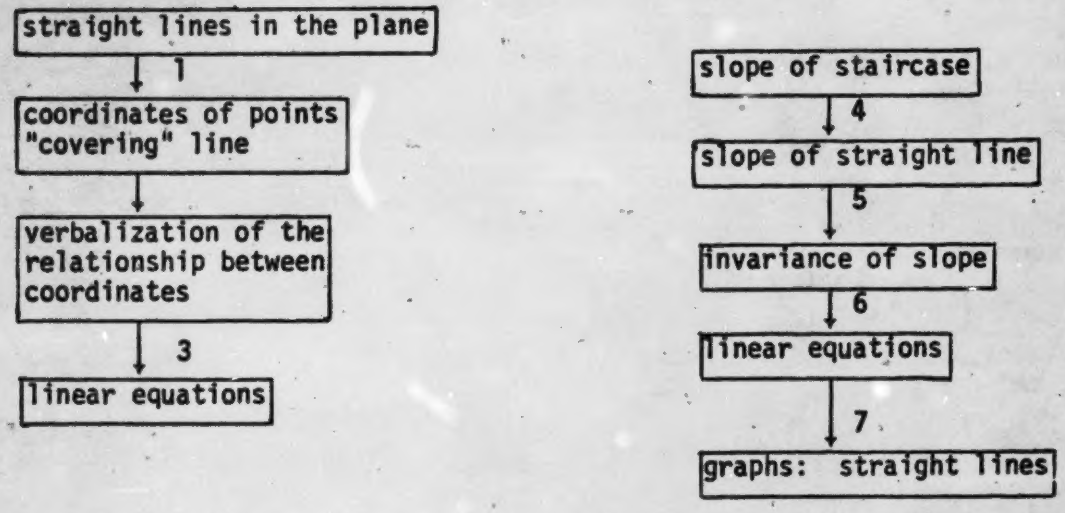
Conclusion

The following diagram illustrates the learning model for the concept of first degree equations in one unknown:



Quite obviously the first step involves the construction of meaning for equations, the new mathematical form, thus leading to a level of formal understanding. The operations on arithmetic identities lead the students to the intuitive rule "what you do to one side, you have to do to the other." Hence, when the time comes to solve equations, the algorithms used can be justified thereby giving the learner a relational understanding. And finally, substituting the solution in the equation for purposes of verification, he recovers the arithmetic identity.

The learning model for linear equations in two variables is more complex and can be summarized as follows:


straight lines in the plane
1
coordinates of points "covering" line
verbalization of the relationship between coordinates
3
linear equations
slope of staircase
4
slope of straight line
5
invariance of slope
6
linear equations
7
graphs: straight lines

The perception of a straight line in the plane can be viewed as a form of intuitive understanding. "Covering" the line with points allows the student to discern a number pattern relating their coordinates. Pattern recognition alternates between intuitive thinking (guessing) and analytic thinking (verification). By expressing the relationship between the coordinates in the form of a linear equation, the student has thereby attained a level of understanding which is both formal and relational. It is formal in the sense that he has constructed a meaning for a new mathematical form, and relational in the sense that this meaning expresses the relationship between the coordinates of all the points on his straight line. Although this first phase is adequate to construct meaning for the linear equation, it proves to be very inefficient for deriving the equation of straight lines with rational slopes and intercepts and, better methods are needed.

The students' intuitive conception of the slope of a staircase, drawn in a Cartesian plane, corresponds to the angle it makes with the vertical axis. Defining slope as the ratio of the riser to the tread represents a formal understanding of this concept. That the slope of a straight line can be perceived in terms of a staircase attached to it is intuitive. If the student can prove the invariance of slope using similar triangles he then possesses a formal understanding of this notion. Of course, he can arrive at the same conclusion inductively by experimenting and taking different measurements giving him an intuitive understanding. Expressing this invariance as $\frac{y - y_1}{x - x_1} = m$ corresponds to a formal understanding leading to the justification of the linear form.

That ordered pairs can be derived from a linear equation is not necessarily obvious to the student. Substituting a value for one of the variables, evaluating the second variable, and arranging these numbers in an ordered pair are instrumental exercises. The realization that an infinite set of ordered pairs can be obtained remains at the intuitive level. Since by now the student is aware that an ordered pair of numbers determines a point in the plane uniquely, the transfer from ordered pairs to points in the plane can be described as relational understanding. The colinearity of these points will be perceived intuitively by the student until he has experienced graphing nonlinear equations which will provide him with the contrast needed to understand the concept of graph relationally. Nevertheless, by following this learning scheme the student realizes that each straight line can be described by a linear equation and conversely, each linear equation can be represented by its graph. This relationship between two different mathematical forms can be qualified as relational understanding.

As can be seen from this analysis, the elaboration of such learning schemes requires a conscious identification and integration of the different modes of understanding. These learning schemes are by no means unique, and any creative teacher can design new ones. The ones reported here have been developed in "teaching experiments" involving several students and will be presented elsewhere in greater detail. Initially, each step needs to be tested with students in clinical interviews, for many learning schemes may look good on paper but will not correspond to the learner's thinking.

The tetrahedral model has proved to be useful, particularly in its identification of formal understanding resulting from the distinction between mathematical form and mathematical content. Much too often, teaching schemes start with mathematical forms new to the student and ignore the cognitive difficulties he may encounter in the construction of meaning for new definitions, new symbols, and new operations on these symbols.

Didactic reversal emphasizes the need to develop teaching schemes which start by gradually and explicitly constructing meaning for the new form on the basis of the student's existing knowledge thus establishing some continuity in the expansion of mathematical content before this content is expressed in a new form (Note 8). This allows the student to participate in the construction of the new form which now amounts to capping a known mathematical idea with conventional symbols. It is only when such a formal understanding has been achieved that the student is ready for the reverse process, that of finding old and known representations for the new form. Although the student could possibly learn the different representations instrumentally, this is much less likely to occur. For, as Skemp and Mellin-Olsen have observed (Note 1), the student's understanding reflects the way he has been taught. An integrated presentation may lead to an integration of the four modes of understanding.

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ABSTRACTION, GENERALIZATION, AND THE DESIGN OF MATHEMATICAL EXPERIENCES FOR CHILDREN

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An understanding of variables affecting conceptual development is essential to persons constructing environmental situations within which children will learn mathematics. The vast majority of in-school mathematical experiences emanate from scope and sequence considerations and reflect little consideration of psychological variables which have been hypothesized to affect human learning. "Use manipulative materials," "get the child involved," "provide real problem solving situations," etc., are cliches which all support, but few seriously implement at the classroom level. Cognitively oriented psychologists have re-emphasized the role which process considerations might play in the design of learning experiences for children. The current clamor for accountability and in some cases even the generation of a list of minimum mathematical competencies required for high school graduation represent in our minds, a reassertion of the primary concern for product outcomes in the mathematics curriculum. In some ways this is a refutation of the growing acceptance of process oriented objectives which has characterized the last decade. It seems that the parameters of children's mathematical involvements, both process and product, are determined externally with little or no direct consideration for the psychological aspects of human learning.

Researchers are amassing currently a rather impressive data base in support of the use of a variety of manipulative materials in the mathematics program (Suydam, 1976). To ask whether children should use physical materials in learning mathematics is analogous to asking whether children learn most effectively from direct experiences or through pseudo experiences. Evidence exists to suggest that the latter type experiences are characterized by symbol manipulation. These simply have been unsatisfactory.

What conditions are responsible for the current state of affairs? Surely many variables have been contributory, and it would be impossible to enumerate them. It is our contention that one of the most powerful factors is a fundamental incompatability between the nature of the learner and the instructional packaging of the mathematics to be learned.

Several of the models presented in this paper were developed by graduate students in Mathematics Education at the University of Minnesota. The authors are indebted to the following students for their ideational contributions to this paper.
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In general it is fair to say that there exists a disparity between learning theory and instructional design. Most attention seems directed at the decision to include or not include a particular topic, but considerations as to the appropriate mode of instruction to be utilized are also essential.

Assuming a decision to include a particular topic in the mathematics curriculum, the next consideration for the curriculum developer is normally concerned with the mathematical development of that idea (sequencing, settings, practice exercise, etc.). Such concerns are obviously legitimate and necessary, but they are not sufficient. If attention to instructional design terminates with content structuring and sequencing, psychological considerations tend to be deleted from planning. In the final analysis, "how" children learn mathematics may be more important than "what" mathematics they do learn.

Looking at this issue from a slightly different perspective the broad realm of instructional design can be thought to have two broad types of subcomponents or subareas of concern:

a) Extrinsic or pedagogical, i.e.,those issues dealing with the teaching of mathematics and

b) Intrinsic, i.e., those issues dealing with the learning of mathematics.

Clearly, these are two separate yet highly related phenomena. The first of these emphasizes the teacher's role in the instructional process, the second emphasizes the instructional sequence from the learner's point of view. The former tends to be extrinsic to the learner, the latter intrinsic. Many attempts to redesign existing mathematics programs or develop new curricula have failed to provide overt recognition to the learning process. Rather, concern has been primarily with the teaching process. Questions such as the following need much more attention: What are the psychological variables hypothesized to promote conceptual development in children? In adolescents? In adults? How can they be mobilized, utilized, or exploited so as to promote more effective learning? Are these variables different across ages? If so, how? Do such variables differ across particular concepts? Are they unique within concepts? etc., etc., etc. And perhaps most importantly how might (should) these variables be accounted for in the construction and sequencing of activities for children?

The answers to such questions surely would be useful in future attempts to design instructional activities which do consider both the extrinsic and intrinsic aspects of human learning. The model described in this paper deals with the development of embodiment matrices for various mathematical concepts. It is an attempt to systematically provide for certain psychological aspects of conceptual development in the design of mathematical activities for children. In particular, the model is concerned with abstraction and generalization. It is viewed as a bridge between certain psychological aspects of conceptual development and the design and implementation of instructional activities.

The remainder of this paper is divided into five sections

I. Theoretical Foundations
II. An Overview of Related Research
III. General Procedures for General Model Development
IV. Exemplars
V. Additional Thoughts

Theoretical Foundations

The theoretical foundations of these ideas have been provided by the writings of Z. P. Dienes and E. W. Golding (1971a, 1967, 1971b).

Dienes, while generally espousing the views of Jean Piaget, has made contributions to the cognitive psychological view of mathematics learning that are distinctly his own. It is appropriate here to review briefly the major components of the theory of mathematics learning proposed by Dienes. His four "principles" are meant to identify necessary--although perhaps not sufficient, since specific teacher methodology is not considered--criteria for effective mathematics learning.

The Dynamic Principle

The dynamic principle suggests that true understanding of a new concept is an evolutionary process involving the learner in three temporally ordered stages.

1. The preliminary, or "play" stage, is in evidence when the learner is concerned with activities of a relatively unstructured nature but which provide actual student experience to which later experiences can be related. Although unstructured, the activities here are not random. The child is presented with an "educational ball park," within which the concept(s) are embedded, and given complete freedom to "play" in it. The learner is therefore exposed to basic rudiments of concepts in a very informal manner. Since freedom to experiment is essential, the learners' responses, varied though they may be, must be relatively uninhibited by the teacher and treated with genuine respect.

2. In the "becoming aware" stage, which follows the informal exposure afforded by the play stage, more structured activities are appropriate. It is here that the child is given experiences structurally similar to the concepts to be learned.

3. The third component of the dynamic principle is characterized by the emergence of a mathematical concept with ample provision for reapplication to appropriate environmental stimuli. The concept is not considered to be fully operational until it can be freely recognized and applied to relevant situations. Ideally this practice stage will serve a dual role: to solidify the newly formed concept in the child's experience and to serve as a play stage for the next concept to be learned. A cyclical pattern emerges which can be depicted as shown in Figure 1. The completion of this cycle is necessary before a mathematical concept

becomes operational for the learner. Furthermore, Dienes maintains that this cycle is repeated continuously as new mathematical concepts are formulated. The similarity between Dienes' model of learning and Piaget's description of the processes of assimilation and accommodation is apparent.

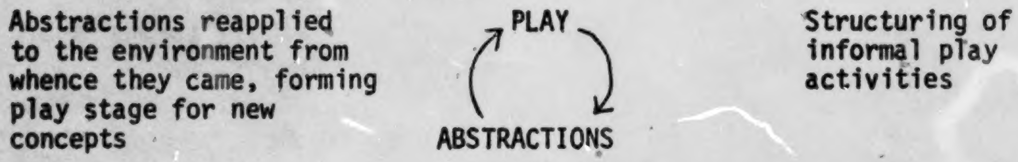

Abstractions reapplied to the environment from whence they came, forming play stage for new concepts
PLAY
ABSTRACTIONS
Structuring of informal play activities

Figure 1. Third Component of Dynamic Principle.

Dienes (1967, 1971) has elaborated on this process and referred to it as a learning cycle. The cycle has been subdivided into six consecutive stages, each of which is an essential component of effective mathematics learning. Briefly summarized, they are as follows:

1. Interaction with the environment.

2. Rule construction of manipulation. This stage is a natural consequence of discovering regularities and of subsequent experimenting with new-found rules of constraints.

3. Comparison of several games or activities having identical structures or rules. (This may be referred to as the search for isomorphisms.)

4. Representation of like (or isomorphic) structures. This involves realization of the "sameness" of the structures in question, together with an ability to determine a method of representing all games (activities) having a similar structure.

5. Symbolization. This is characterized by the investigation of the properties identified in 4. At this level, such investigation is not dependent on the real world referent although it should be available to corroborate findings generated in symbolic form.

6. Formalization. This means the derivation of other properties of the system from those already identified. This is the process of generating theorems (statements logically deduced) of a system from its axioms (rules of the game or self-evident truths). The process of theorem derivation is known as a proof and is considered to be the quintessence of mathematical activity.

These stages describe the general characteristics of a sequence of experiences that result in the appropriate development and subsequent abstraction of a given concept.

The remaining components of Dienes' theory are in a sense contained within the dynamic principle and should be considered to exist within the framework already established.

The Perceptual Variability Principle (multiple embodiment principle)

In an attempt to make provision for individualized learning styles and at the same time promote abstraction of mathematical ideas, Dienes suggests the perceptual variability principle as an indispensible element in the process of concept formation. This principle suggests that conceptual learning is maximized when children are exposed to a concept in a variety of physical contexts or "suits of clothes." The experiences provided should differ in outward appearance while retaining the same basic conceptual structure. Children can become sidetracked with irrelevant characteristics of a situation, especially when their grasp of the concept is incomplete. The Association of Teachers of Mathematics (1967) has captured the essence of this phenomena in the following quote:

> As adults, with some knowledge, our approach to many things is abstract. We readily ignore some characteristics in order to be satisfied that the remaining properties will do something for us ... Children do not ignore the different characteristics of materials so readily. In fact, we could say that the ignoring of different characteristics is directly proportional to the degree of knowledge one already has or is expected to have. This means that for any given situation which may seem to be a reasonable starting point for a discussion, the attention the teacher pays it and the attention the children pay it will be different and this can give rise to difficulties in comprehension--each misunderstanding the other (p. 168).

Dienes believes that the current mode of mathematics instruction promotes learning which is associative in nature. Children are encouraged to associate a particular mathematical process or operation with a particular situation. A "bag of tricks" approach is often used by the teacher, applying established mathematical procedures in a manner which tends to be routine or habitual. The problem with associative learning arises when the student finds himself in a situation for which he does not possess a ready-made response pattern. Under these conditions the associative learner becomes bewildered and unable to abstract relevant components of the problem situation. It is, therefore, unlikely that he will be able to reconstruct the problem in a manner which will ultimately lead to a solution. As an alternative to this type of learning, Dienes suggests an instructional setting which would promote abstraction rather than association. (Abstraction is here defined as the ability to perceive a concept irrespective of its concrete embodiment.) Dienes believes that by providing children with the opportunity to see a concept in different ways and under different conditions, the purposes of promoting abstraction will be more adequately served.

The Mathematical Variability Principle

Similar to the Perceptual Variability Principle in that it encourages multiplicity in patterns of exposure, the mathematical variability principle asserts that if a mathematical concept is dependent upon a certain number of variables (quantities which may assume a variety of values), then variation of these is an important prerequisite for the effective learning of the concept. For example, a parallelogram is defined as a quadrilateral having its opposite sides parallel. Although its shape can be varied by changing the length of its sides and the size of its angles can be altered, the only crucial factor for a quadrilateral to remain a parallelogram is that its opposite sides remain parallel. If a child were exposed to only those parallelograms with equal angular measure or only to those which have constant proportion relative to the length of its adjacent sides, surely he would not develop a general concept of parallelogram. The mathematical variability principle suggests that in order to maximize the generalizability of a mathematical concept, as many irrelevant mathematical variables as possible (in this example, the size of angle and the length of side) should be varied while at the same time keeping the relevant variables (opposite sides parallel) intact.

By the same token, if one is interested in promoting an understanding of place value, it is desirable to vary the base while providing experiences which highlight the consistency of regrouping procedures, the importance of relative positional value, and the appropriate way to record results. When implemented, the mathematical variability principle encourages the student to separate the "wheat from the chaff" by systematically isolating relevant variables in the consideration of a given concept. This could result in conceptual learning with a degree of precision transcending that usually found in the mathematics classroom.

The Constructivity Principle

Here Dienes identifies two kinds of thinkers: the constructive thinker and the analytical thinker. He roughly equates the constructive thinker with Piaget's concrete operational stage and the analytical thinker with Piaget's formal operational stage of cognitive development. This principle states simply that "construction should always precede analysis." It is analogous to the assertion that children should be allowed to develop their concepts in a global intuitive manner emanating from their own experiences. According to Dienes these carefully selected experiences form the cornerstone upon which all mathematics learning is based. At some future point in time, attention should be directed toward the analysis of what has been constructed; however, Dienes points out that it is not possible to analyze what is not there in some concrete form.

Summary and Implications

The unifying theme of these four principles is undoubtedly that of stressing the importance of learning mathematics by means of direct interaction with the environment. Dienes continually asserts that mathematics learning is not a spectator sport and, as such, requires a very active type of physical and mental involvement on the part of the learner. In addition to stressing the environmental role in effective conceptual learning (Dynamic Principle), Dienes in his mathematical and perceptual variability principles addresses the problem of providing for individualized learning rates and learning styles. His constructivity principle aligns itself closely with the work of Piaget and suggests a developmental approach to the learning of mathematics which is temporally ordered to coincide with the various stages of intellectual development.

This paper will attempt to apply the mathematical and perceptual variability principles to specific mathematical concepts. This application will result in the development of a two dimensional matrix for each mathematical concept considered. The vertical and horizontal axes of each matrix will consist of examples of the perceptual and mathematical variables which are relevent to that particular concept. Before suggesting procedures for the development of such specific models it is appropriate to briefly review some of the more important literature in this area.

An Overview of Related Research

The model for the design of instructional activities proposed in this paper relies heavily on the "parallel-series" organization of instructional activities. This arrangement is best characterized by the term multiembodiment, which will provide the central focus for this brief review of related research.

First, it is important that proper embodiments of the mathematical concept be provided. When physical materials are used in instruction, they should provide a concrete representation, or embodiment, of a mathematical principle. When different, yet appropriate, concrete materials are used to develop the same mathematical idea, a "multiple embodiment" is provided. This approach to instruction demands that teachers use a variety of perceputally different materials in developing a mathematical concept. Furthermore, it is essential that the physical embodiments be clear and meaningful to the learner if later symbols associated with these embodiments are to be understood (Fennema, 1972).

One of the basic assumptions underlying the use of physical materials is that pupils learn best through active involvement with concrete experiences. However, this depends on the concepts involved, the age and previous experience of students, and the particular materials supplied. One cannot say, at least with any degree of certainty, that using physical materials with active learning experiences will be effective in helping all children master all types of mathematical objectives. Too much depends on the complex interaction among and between pupils, teachers, and materials. Consequently, research regarding the value of physical materials has been until recently generally inconclusive (Fennema, 1972; Kieren, 1969; Wilkinson, 1974). That is, a case could be made for either side simply by choosing research reports that support a particular bias.

In an unusually comprehensive review and synthesis of current research conducted primarily since 1970, on Activity-Based approaches to mathematics teaching, Suydam and Higgins (1976) have summarized the results of twenty-three studies dealing with the use and non-use of manipulative materials and twenty-eight studies contrasting concrete, pictorial, and symbolic types of materials. In addition, this very useful document summarizes results of many other studies dealing with the interaction of multiple embodiments, age, ability, and material use and with the impact of materials on the learning of specific mathematical content.

The findings summarized in the Suydam and Higgins paper include the following:

1. Approximately half of twenty-three studies considered reported significant differences in achievement in favor of the treatment groups using some form of manipulative material. Ten other studies reported no significant difference. The authors concluded that "lessons using manipulative materials have a higher probability of producing greater mathematics achievement than do non-manipulative lessons" (p. 58).

2. With respect to the use and sequencing of concrete, pictorial, and symbolic activities and their influence on student achievement the authors concluded:

   a) The use of concrete-symbolic produced higher levels of student achievement than pictorial-symbolic or symbolic,

   b) The use of concrete-symbolic produced higher levels of achievement than concrete-pictorial-symbolic which in turn was better than pictorial-symbolic, and

   c) The use of concrete-pictorial was not significantly better than pictorial-symbolic (p. 58).

3. Regarding the use of embodiments, and its impact on conceptual development, Suydam and Higgins suggest that "research attempting to pin down this causal relationship by designing teaching treatments using varying numbers of multiple embodiments is generally inconclusive" (p. 26). They conclude that "a program should not be selected only because it uses multiple embodiments . . . Varying embodiments may aid children in making mathematical applications, but no studies have measured this specific achievement " (p. 59).

4. The use of manipulative materials appears to be important across grade level, achievement level, socio-economic level, and mathematical topic (p. 59-62).

The research reviewed in the Suydam and Higgins document establishes the importance of using manipulative materials to help children learn mathematical concepts. However, the questions of how specific materials should be organized and sequenced, the subject of interest in the present paper remains essentially unanswered.

Several recent reviews of research (Beougher, 1967; Fennema, 1972; Kieren, 1969; Suydam & Weaver, 1970) also have identified investigations related to multiple embodiment. A critical review of published studies reveals a wide range in the quality of research and therefore raises doubts about the credibility of certain findings. Nevertheless, it is clear that the research does not consistently support or refute a multiple-embodiment approach to teaching mathematics. In fact, the one common thread among these studies is that learning mathematics depends more on the teacher than on the embodiment used.

In addition to research specifically related to mathematics education, there are several more general findings which support a multiple-embodiment approach to instruction:

1. Pupils learn differently. Therefore increasing the number of embodiments increases the likelihood of correctly matching an instructional apprcach with a child's learning preferences.

2. Pupils enjoy new and different activities. A change in the physical setting (i.e.,by a new embodiment) is usually accompanied by renewed interest and enthusiasm.

3. Pupils often overgeneralize and therefore incorrectly transfer ideas from one situation to another. Instruction based on a variety of experiences encourages the abstraction of essential ideas that are common to several activities. Consequently the learner is less likely to incorrectly jump to a conclusion in a situation where only one of several necessary ideas is presented.

Although none of these statements deal exclusively with mathematics learning, they are consistent with a multiple-embodiment approach to teaching mathematics.

Many additional issues related to embodiments and levels of concept formation need to be considered. Several questions, related to quality, number, sequence, and time allocation for embodiments, seem particularly important.

For a given mathematical concept:

1. Which embodiments are most effective in fostering concept formation? Are some easier, or more difficult, for pupils to understand? Are some appropriate for slow learners, but not necessarily for faster learners?

2. To how many embodiments should each pupil be exposed? Is one embodiment sufficient? Two? What is the optimal number of embodiments? For which students? Do too many embodiments confuse a slow learner?

3. Does a given set of embodiments have an inherent sequencing which would maximize learning? That is, does the order in which the embodiments are presented make a difference?

4. Does each embodiment require the same amount of time to develop? How much time should be devoted to each embodiment? Should different embodiments be developed together (same day or week), or should they be spread over a period of time (several weeks or months) using a spiral approach?

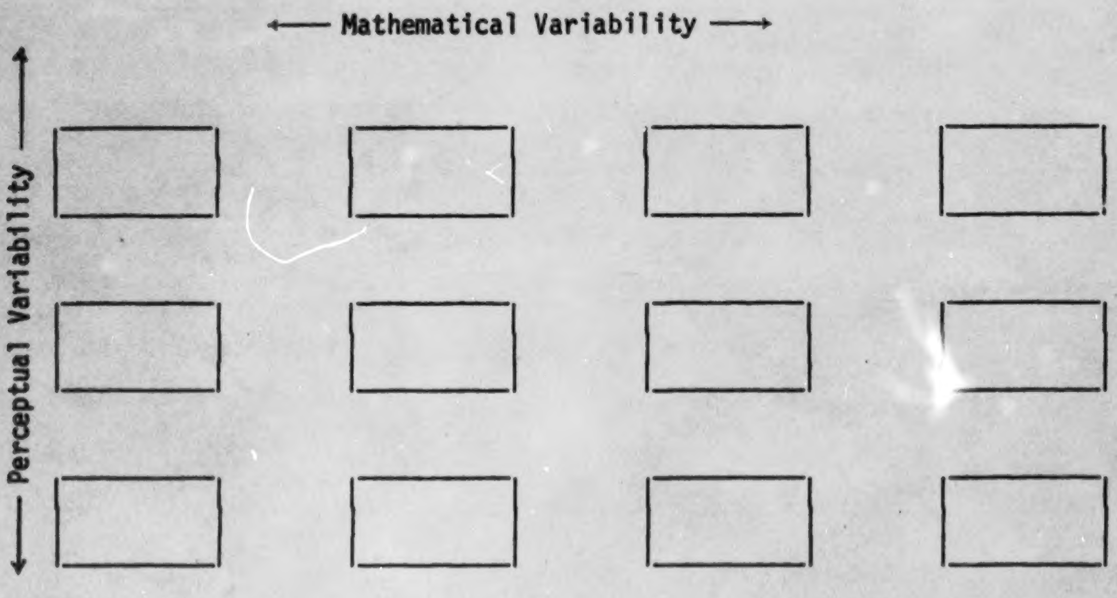
These issues are complex. It seems unlikely that research will provide simple answers. It seems more likely that answers to such questions will need to be qualified in terms of teachers, mathematical content, and embodiments used, as well as by the pupil's ability, background, and achievement level. Nevertheless, these issues are significant.

Future research studies related to multiple-embodiment should be conducted for longer periods of time than the vast majority of the reported studies. Many mathematical concepts simply cannot be learned in three to six weeks. Extended time frames also are crucial if retention as well as skills in generalizing and/or abstracting are to be of major concern.

We turn now to a description and discussion of the general procedures which are used to generate a two dimensional matrix for a particular mathematical concept. This discussion is followed by specific examples of such matrices.

General Procedures for Specific Model Development

All exemplars presented in this paper reflect the two variability principles. The individual examples are topic specific and are not unique. Generally, the specific models are developed in the following manner: The first step is to identify an appropriate mathematical concept. The identification of the mathematical and perceptual variates constitutes the next and perhaps most crucial stage in the development of a particular example. The perceptual and mathematical variates will define the dimensions of a two-way matrix. This can be depicted as:
CONCEPT ______X______



In this example, three "levels" (rows) of the perceptual variate have been identified for the concept in question along with four levels (columns) of the mathematical variate. Each cell in the model defines both a specific physical material (embodiment), and a corresponding level of an appropriate mathematical variate.

Once this has been done, it is then reasonable to begin to synthesize existing activities, lessons, practice sheets, etc. The vast majority of such activities are already in existence. This procedure, therefore, does not require a great deal of curriculum development, only the synthesis of existing software, so as to conform with the parameters established by the identification of appropriate perceptual and mathematical variates. This is a particularly important point since the establishment of large scale curriculum development projects appears to be unlikely at this time. The cell entries can be physically represented by "shoe-box" activities complete with both hardware and software. These shoe boxes might be used at a mathematics station within the classroom. If conveniently stored, students could be responsible for obtaining these shoe boxes, completing the activity within, either individually or in small groups, then returning it for use by other students. Storage of such materials generally has not been a major concern.

Decisions would naturally need to be made as to the extent of student involvement in a particular row or column. Instructional components implied by cells in a matrix will not be experienced always by children in a single (consecutively ordered) series of experiences. It is quite probable that involvement in many mathematical concepts will be spread over several years.

All exemplars will contain a row which deals primarily with symbolic representation and/or manipulation. Symbolic representation is considered to be an "embodiment" of a particular concept even though in and of itself it is devoid of any physical context. Symbolic representation, then, is a different kind of embodiment with special useful properties. As such it is deserving of differential treatment. One aspect of such differential treatment is the fact that it will be regularly juxtaposed with other physical embodiments. Symbolic representation is routinely placed in the bottom line to imply that it is the most complex and that students would not become involved exclusively with the manipulation of symbols unless and until they have had a number of different experiences in less abstract form.

The reader will note that in the examples to follow not all possible "levels" of either the mathematical or perceptual variables have been included. Perhaps a favorite or highly effective embodiment does not appear. Perhaps an obvious or important mathematical variate is absent. Such omissions are acknowledged. It is not the intent to specify "the best" embodiments or levels of the mathematical variate. Rather the point of concern is that the psychological factors (abstraction and generalization) to which the perceptual and mathematical variability principles address themselves are indeed worthy of overt attention and need to be provided for when designing instructional activities for children. The identification and structuring of activities in this manner serves to highlight this overt recognition. One major difficulty in designing the "best" two dimensional matrix for a particular concept which depicts the most effective physical embodiments and the most important mathematical variates is the fact that research in these areas is fragmentary and of uneven quality. That is, the relative effectiveness of various embodiments and the identification of the most important mathematical variates has not been firmly established by empirical research.

It is therefore not possible to identify the "best" matrix with any degree of assurance. The exemplars to follow undoubtedly will strike a familiar cord with most readers. The organizational format perhaps is a bit different but the components of individual cells will not be unfamiliar.

Each has been fitted to the general paradigm already established. The next section of this paper provides specific examples. These are to be read subject to the qualifications noted above.

Exemplars

The first example (see Figure 2) deals with the concept of place value. In this instance, the selection of appropriate perceptual and mathematical variates is fairly clear. The concept of "base" or "number system" constitutes the mathematical variable to be systematically manipulated. This is appropriate since operations within all number bases are governed by the same basic regrouping procedures except that the number of pieces involved in a trade will vary with the number base

CONCEPT: PLACE VALUE

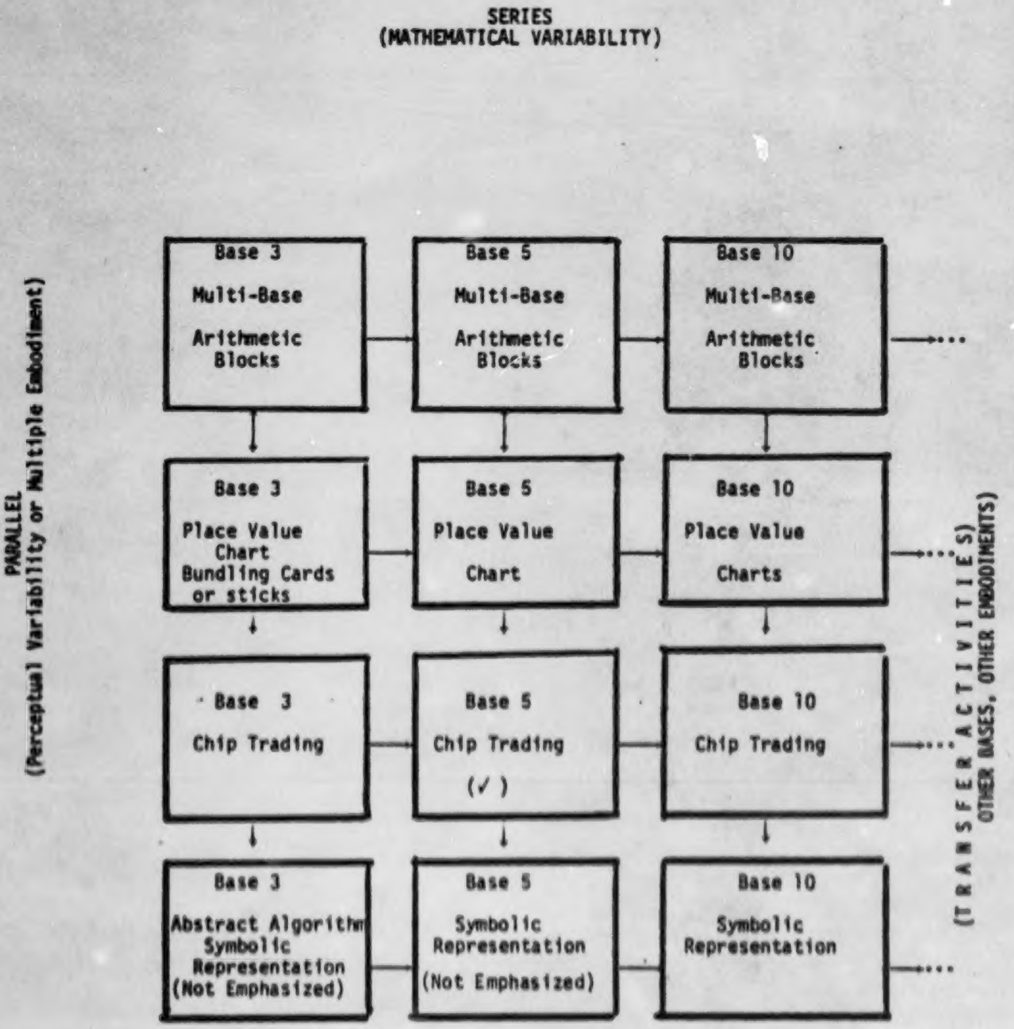

SERIES
(MATHEMATICAL VARIABILITY)
PARALLEL
(Perceptual Variability or Multiple Embodiment)
Base 3
Multi-Base
Arithmetic
Blocks
Base 5
Multi-Base
Arithmetic
Blocks
Base 10
Multi-Base
Arithmetic
Blocks
Base 3
Place Value
Chart
Bundling Cards
or sticks
Base 5
Place Value
Chart
Base 10
Place Value
Charts
Base 3
Chip Trading
Base 5
Chip Trading
(✓)
Base 10
Chip Trading
Base 3
Abstract Algorithm
Symbolic
Representation
(Not Emphasized)
Base 5
Symbolic
Representation
(Not Emphasized)
Base 10
Symbolic
Representation
(TRANSFER ACTIVITIES)
OTHER BASES, OTHER EMBODIMENTS)

Figure 2

considered. When students realize that the "rules" governing regrouping procedures are consistent across bases they have in effect generalized the concept, that is, they have developed the ability to apply these rules to any arbitrary system.

Within each base it is also possible to employ a wide variety of physical materials. Children should experience the concept while working with a number of materials that are different in appearance but similar in structure or function. The various materials thus become the perceptual variates or embodiments. When students realize that multibase arithmetic blocks, place value charts (using wooden sticks), and chip trading activities can be utilized in ways which are structurally similar, i.e. can all be used to illustrate the same basic mathematical concept, then they will have abstracted the concept from its physical context. It is not the ability to work with any specific set of materials which is the abstractive component of this mathematical concept, but rather, a realization of their sameness. Obviously if such perceptual variability is not provided within the instructional sequence then opportunities for such abstraction are minimized.

One possible arrangement of learning activities concerned with the concept of place value is also depicted by the matrix in Figure 2. Each cell implies a collection of activities dealing with a particular base and a particular type of manipulative material. Unidirectional arrows indicate temporal sequencing. Dual directional arrows imply that interchangeable sequencing is appropriate. Note that in this case work with the "easier" bases is to precede student interaction with base 10. Also note that symbolic representation without some form of concrete referent is considered last within a particular base, although it might be juxtaposed with specific materials earlier in the sequence.

Such ordering is consistent with the Dienes' constructivity principle since it does not place children in a position where they are asked (or expected) to analyze in the abstract something that does not mentally exist in some type of concrete form to begin with. In short, students do not work exclusively with symbols unless and until they have had experience with the concept at the enactive (concrete-manipulative) level of operational thought.

Certainly all students would not be expected to complete activities within all matrix components. Some students will need more exposure to the concept than others; however, all students probably should sample from two or more rows and two or more columns if appropriate conceptual abstraction (promoted by perceptual variability) and generalization (promoted by mathematical variability) are to occur. The ultimate decision as to the number of matrix components considered by a given student is, and should be, left to the teacher. Hopefully research will eventually aid such decisions. Such decisions should be made, with full acknowledgement of the fact that premature student abstraction will only result in future difficulties. Furthermore, these decisions should be made on an individual rather than on a large group basis.

Each cell in the matrix implies both a mathematical and a perceptual variate, in this case a number base and a specific type of manipulative material respectively. For example, the "contents" of the checked cell (✓) in Figure 2 would consist of a number of activities dealing with the chip trading kit in a "trade for 5" setting.

Research is currently underway on this particular model to assess the impact of the variability principles (alone and in concert) on conceptual development with primary grade children.

Area

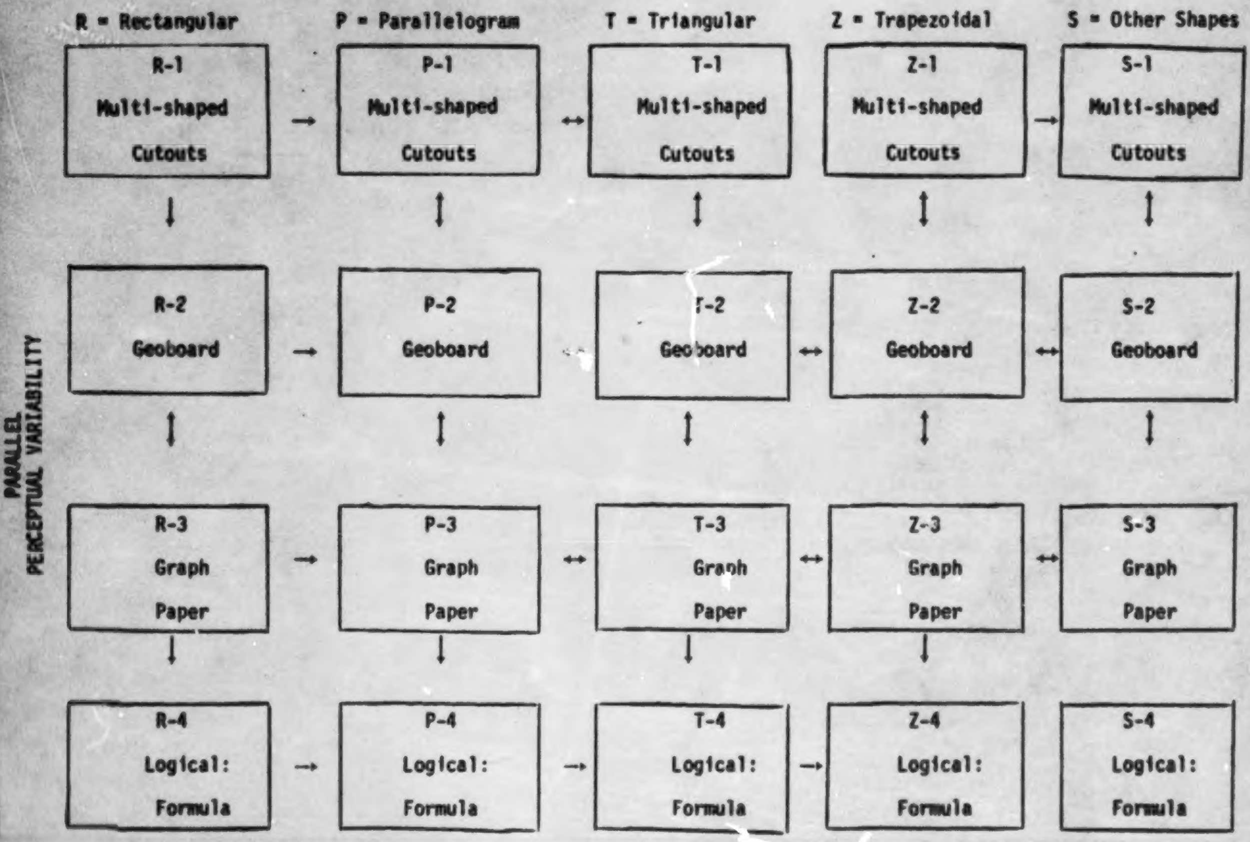
Figure 3 illustrates perceptual and mathematical variability dimensions as applied to the concept of area. In this particular model, area is represented by individual tiles, as an enclosed space on a geoboard and as an enclosed space on a piece of graph paper. In addition, of course, abstract formulas are included as the most sophisticated interpretation of the concept. Shape has been identified here as the mathematical variable to be systematically manipulated. As can be seen, the shapes manipulated include four which lend themselves to simple formulas and the fifth (nonregular shapes) for which formulas are not readily apparent. Children can find areas of physical spaces using the technique of superposition, using for example floor tiles at a very early age. Certainly one would not expect these same children to become involved in abstract formulas for shapes of a trapezoidal nature. Furthermore, many of the skills developed here within various embodiments will be called upon later to determine surface area of geometric solids.

The arrows in Figure 3 indicate the order in which activities could (or should) be presented. Noteworthy is the unidirectional arrow between the concrete embodiments and the logical formulas. Once again these arrows reassert the constructivity principle as suggested by Dienes, that is, one cannot analyze abstractly what has not been previously constructed in a more concrete form. Note also the bidirectional arrows connecting other matrix cells. Such directionality implies considerable flexibility in the selection of activities. It would be possible to debate the implications of the placement of several of these bidirectional arrows, however, for our purposes here we wish only to suggest that "absolute" sequences probably do not exist. In fact, discussion as to the appropriateness of the relative position of any column or row would no doubt be a fruitful activity for the teacher planning to use such an organizational format.

Extending the Natural Numbers to Integers

Figure 4 suggests ways of expanding the concept of number to include negative integers. Five different "physical" (or conceptual) embodiments are illustrated. In each case identification of the mathematical variable was limited to three components. Undoubtedly this particular concept has many different manifestations. The intent, however, is not to argue for the uniqueness of appropriate mathematical variates, but rather, to suggest a mental set for teaching particular concepts which systematically manipulates those variables not fundamental to the concept under consideration. In fact, particular rows, columns, and/or cells might themselves be

CONCEPT: AREA
SERIES
MATHEMATICAL VARIABILITY: (SHAPE)


R = Rectangular
P = Parallelogram
T = Triangular
Z = Trapezoidal
S = Other Shapes
R-1 Multi-shaped Cutouts
P-1 Multi-shaped Cutouts
T-1 Multi-shaped Cutouts
Z-1 Multi-shaped Cutouts
S-1 Multi-shaped Cutouts
R-2 Geoboard
P-2 Geoboard
T-2 Geoboard
Z-2 Geoboard
S-2 Geoboard
R-3 Graph Paper
P-3 Graph Paper
T-3 Graph Paper
Z-3 Graph Paper
S-3 Graph Paper
R-4 Logical: Formula
P-4 Logical: Formula
T-4 Logical: Formula
Z-4 Logical: Formula
S-4 Logical: Formula
PARALLEL
PERCEPTUAL VARIABILITY

Lesson should be taught in order which arrow appears
Lesson could be presented either order

Figure 3

conceptualized as a concept to be taught. In this event, perceptual and mathematical variables might then be identified for that particular subunit of a given model. For example, in Figure 4 the concept of addition which is identified here in the second column clearly can be identified as a concept in and of itself. Perceptual and mathematical variates appropriate to the concept of addition (or subtraction or counting and ordering) would then be identified. Were one to consider the concept of addition it is likely that Cuisenaire rods, counters, number lines, etc. as well as symbolic algorithms would surface as perceptual variates. Appropriate mathematical variates might be identified also. The fact that the model(s) suggested can be continually atomized is simply a reassertion of the sequential nature of mathematics. Clearly a child should not be involved in the activities implied by Figure 4 if that child does not have at least a passing acquaintance with the concept of addition of natural numbers.

It is not necessary here to elaborate on the perceptual variables identified in Figure 4. Suffice it to say that all provide a slightly different physical embodiment of the identified concept. Such breadth of exposure should limit the probability of associating the concept with one particular situation. Such overt variation is hypothesized to maximize the abstractive potential of the instructional sequence. As with previous models it also should be noted that the row entitled Symbolic Representation in reality will be juxtaposed with other embodiments. Symbolic representation is placed routinely at the bottom line to imply that students would not become involved solely with the manipulation of symbols before having had a number of different experiences in a less abstract form.

Figure 5 is concerned with: "Equivalent Fractions." In this case the identification of the perceptual and mathematical variates is clear and further discussion does not seem necessary.

Figure 6, dealing with the Concept of Measurement, is at once different from the previously mentioned models. Originally proposed by two especially innovative graduate students (Koch and Garfield), Figure 6 adds a third dimension; this dimension attempts to impose a number of general process or methodological considerations on the structure already established. The implications of the addition of a process dimension is intriguing. At this point, however, the number of additional cells required because of the addition of a third dimension seems unwieldy. Many cell components would require the creation of, rather than the synthesis of existing software. Reality constraints suggest that initially a more conservative approach utilizing only two dimensions be considered. Ultimately the process issue must be addressed. Whether the components of this third dimension are the ones suggested in Figure 6 is less important than the fact that process be considered overtly at all. Our reluctance to develop this idea further is due only to the recognition that large scale curriculum development is a major undertaking and as such transcends the scope of the suggestions made in this paper.

CONCEPT: EXTENDING THE NATURAL NUMBERS TO INTEGERS

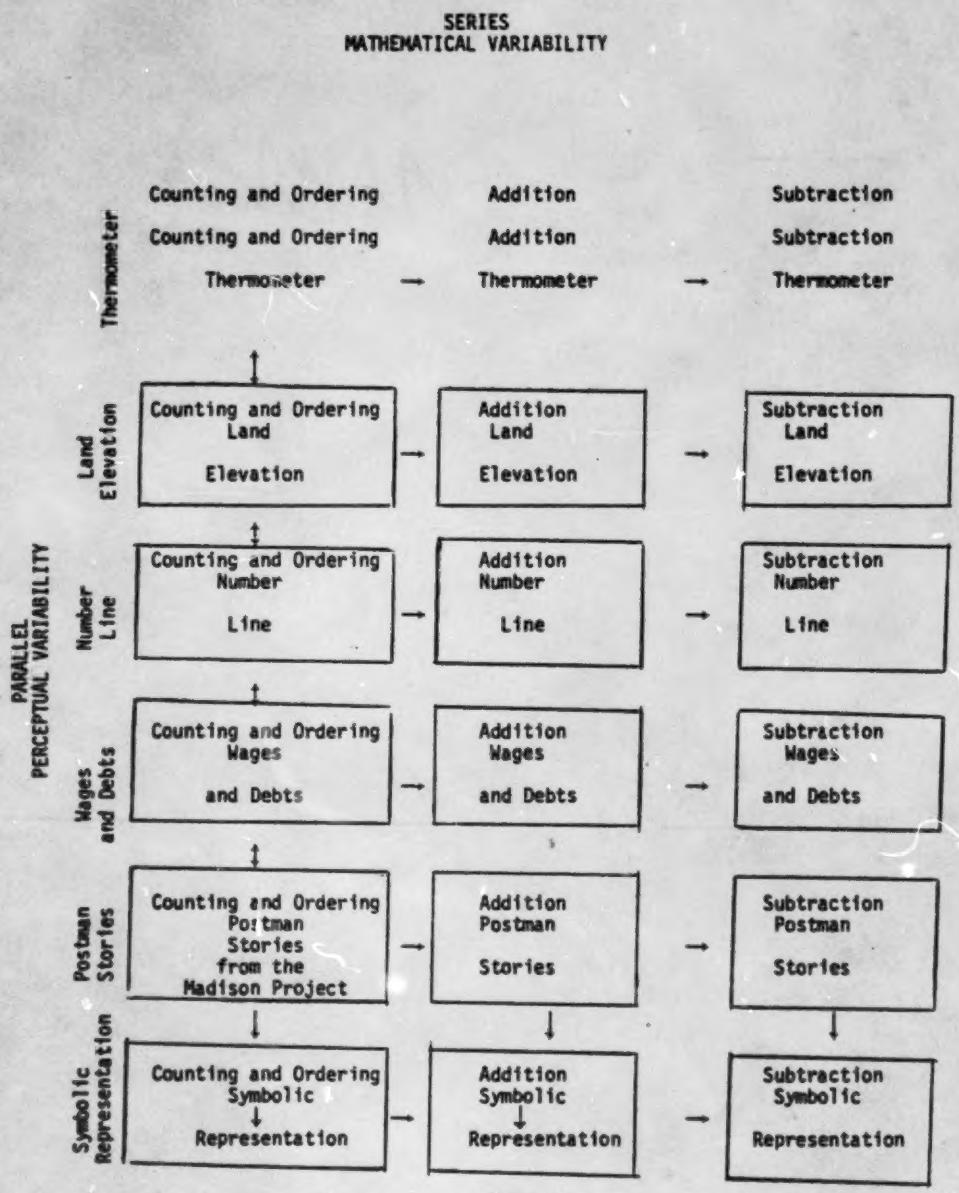

SERIES
MATHEMATICAL VARIABILITY
PARALLEL
PERCEPTUAL VARIABILITY
Counting and Ordering
Addition
Subtraction
Thermometer
Counting and Ordering Thermometer
Addition Thermometer
Subtraction Thermometer
Land Elevation
Counting and Ordering Land Elevation
Addition Land Elevation
Subtraction Land Elevation
Number Line
Counting and Ordering Number Line
Addition Number Line
Subtraction Number Line
Wages and Debts
Counting and Ordering Wages and Debts
Addition Wages and Debts
Subtraction Wages and Debts
Postman Stories
Counting and Ordering Postman Stories from the Madison Project
Addition Postman Stories
Subtraction Postman Stories
Symbolic Representation
Counting and Ordering Symbolic Representation
Addition Symbolic Representation
Subtraction Symbolic Representation

Figure 4

CONCEPT: EQUIVALENT FRACTIONS
SERIES
MATHEMATICAL VARIABILITY

Common Fraction
form a/b:
a, b natural numbers

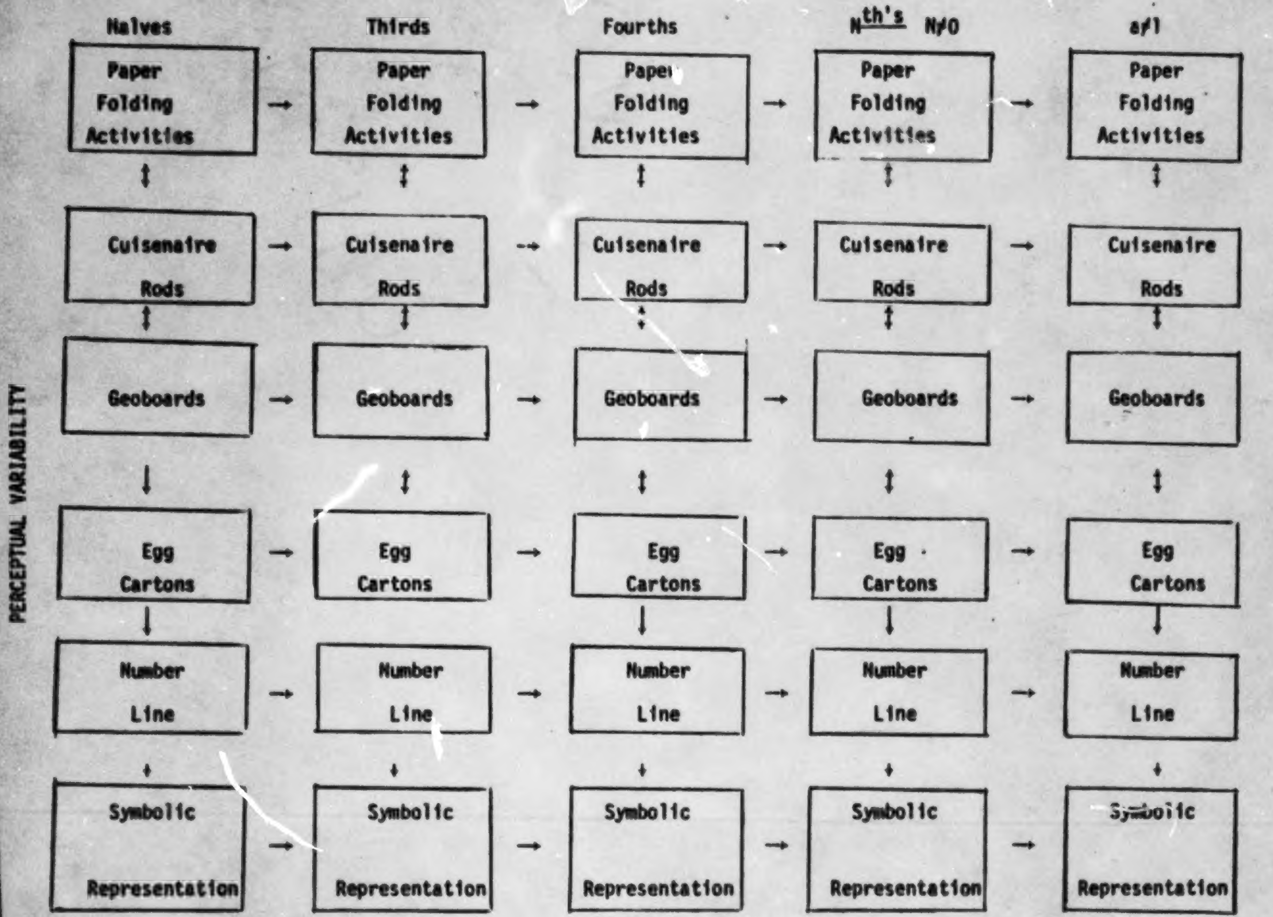
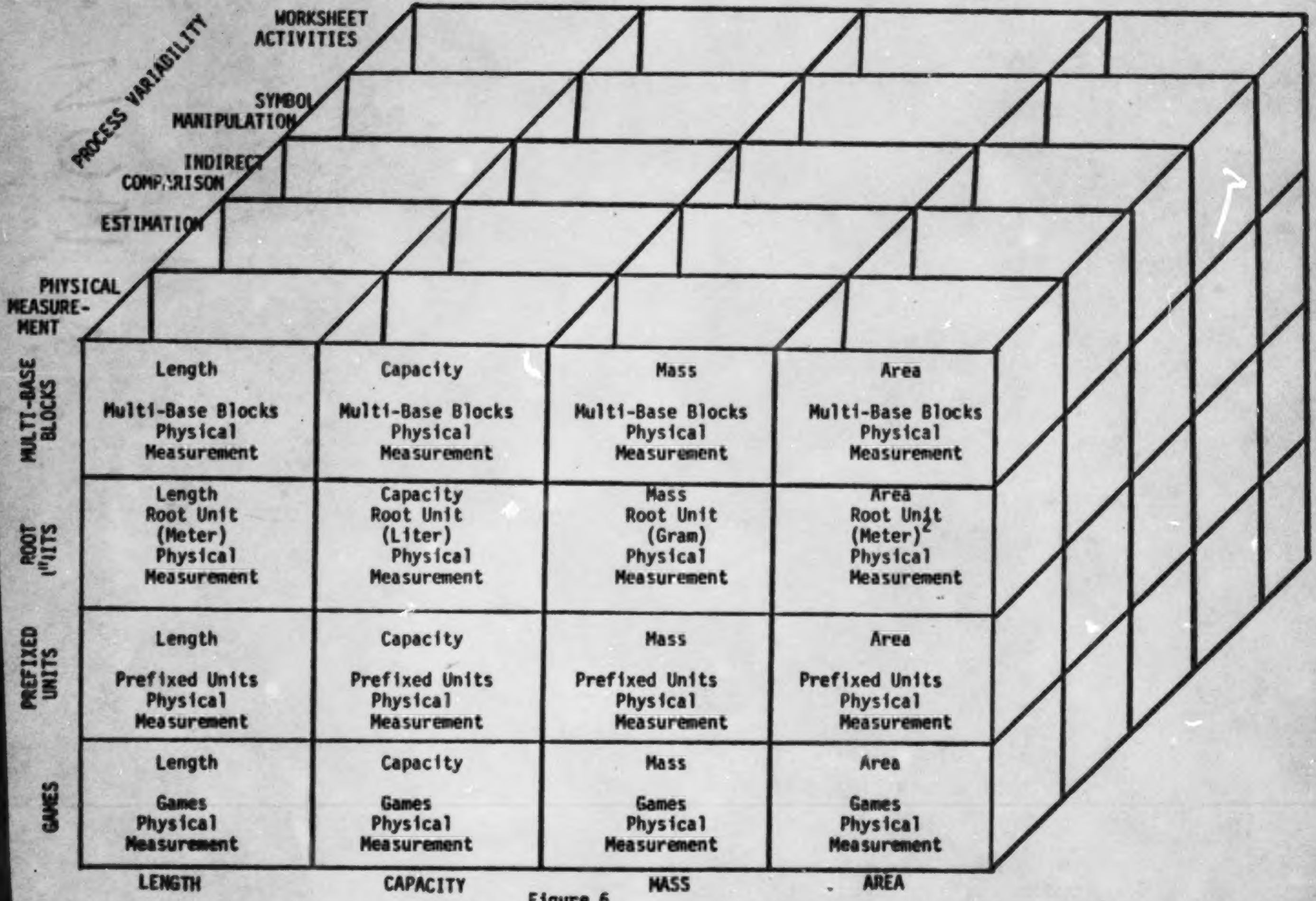

Halves
Thirds
Fourths
Nth's N≠0
a≠1
PERCEPTUAL VARIABILITY
Paper Folding Activities
Cuisenaire Rods
Geoboards
Egg Cartons
Number Line
Symbolic Representation

Figure 5

CONCEPT: MEASUREMENT

SERIES
MATHEMATICAL VARIABILITY


PROCESS VARIABILITY
WORKSHEET ACTIVITIES
SYMBOL MANIPULATION
INDIRECT COMPARISON
ESTIMATION
PHYSICAL MEASUREMENT

	LENGTH	CAPACITY	MASS	AREA
MULTI-BASE BLOCKS	Length Multi-Base Blocks Physical Measurement	Capacity Multi-Base Blocks Physical Measurement	Mass Multi-Base Blocks Physical Measurement	Area Multi-Base Blocks Physical Measurement
ROOT UNITS	Length Root Unit (Meter) Physical Measurement	Capacity Root Unit (Liter) Physical Measurement	Mass Root Unit (Gram) Physical Measurement	Area Root Unit (Meter)^2 Physical Measurement
PREFIXED UNITS	Length Prefixed Units Physical Measurement	Capacity Prefixed Units Physical Measurement	Mass Prefixed Units Physical Measurement	Area Prefixed Units Physical Measurement
GAMES	Length Games Physical Measurement	Capacity Games Physical Measurement	Mass Games Physical Measurement	Area Games Physical Measurement

Figure 6

Additional Thoughts

The non-uniqueness of the perceptual and mathematical variables utilized in these illustrations has been noted. Certainly some degree of discretion, professional judgment, and forethought is advised in their selection; but in the final analysis they are clearly not unique. This condition was at first disturbing to us. We naturally wanted the models developed to be "the right ones" for the concepts under consideration. When we found ourselves continually faced with alternative interpretations and/or possibilities, we felt a strong need to make definite decisions as to the inclusion of one or another embodiment or mathematical variate. It was eventually realized that there probably is no single best way to get "from here to there" and that a variety of routes are possible.

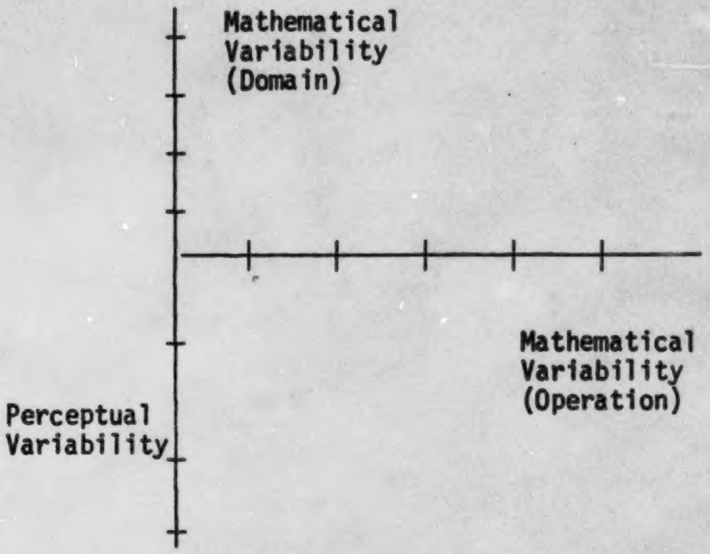
The ultimate potential in the paradigm suggested here then is surely not in its ability to define once and for all time THE manipulatives (perceptual variates) to be utilized or THE mathematical aspects of a concept which are in need of systematic variation. Its value stems from its use in reminding curriculum developers and curriculum implementers (teachers) that psychological factors should be attended to consciously when attempting to construct and implement meaningful activities for children. Abstraction and generalization are not new terms to persons concerned with the general field of conceptual development. Clearly they are relevant to mathematical concepts as well. It seems reasonable to suggest that a child has truly learned a mathematical concept only when s(he) is able to abstract it from its physical context and generalize it to new situations. The variability principles as originally suggested by Dienes have been adapted from the psychological literature so as to relate more directly to the abstraction and generalization of mathematical concepts.

Organizing instructional program components along the lines of the two-dimensional matrix suggested herein does not involve a complete redefinition of the ideas involved, nor does it imply the need for a vast amount of curriculum development. Individual cell components for any conceivable matrix already exist. What would be required, however, is a re-thinking, a re-sequencing, and perhaps some degree of re-structuring of those activities. The synthesis of relevant cell components for a given concept is not a monumental task given 1) the selection of appropriate mathematical and perceptual variables and 2) a collection of activities from existing texts, collections of assignment cards, project materials, or other sources.

It also must be noted that instructional design matrices of this type will not always be rectangular; that is to say, for some concepts, appropriate mathematical variates might not exist for each and every perceptual variate. Such a situation would result in an imbalanced matrix. We do not believe this to be a problem.

Dienes is not particularily specific in defining the necessary and sufficient conditions for identifying the appropriate mathematical and perceptual variates for a given concept.

The concept of perceptual variation has been less elusive than the concept of mathematical variation. Perhaps this is because the latter can be interpreted in terms of its domain, i.e.,the mathematical characteristics inherent in the concept itself or in terms of operation, i.e., the mathematical and/or arithmetical procedures which can be carried out over the concept in question. It is not clear which of these interpretations should be used in deciding upon appropriate mathematical variates. It is possible that these should not (must not?) be mutually exclusive and that both need simultaneous consideration. If true, a three-dimensional axis[1] would more appropriately fit this expanded conceptualization, i.e.,



Additional thinking must be done on this important issue. Impressions at this time suggest that some combination of both is probably optional; that is, both interpretations need to be considered when identifying the mathematical variate for a given concept.

It has been our experience that once the perceptual and mathematical variates of a concept have been identified, our thinking on the concept has been substantially clarified.

The actual construction of such models can be a valuable experience for persons concerned with the teaching of mathematics to children. Preliminary indications resulting from our work with graduate students seem to support this conjecture.

Lastly, we must emphasize the need for systematic research on the impact of these variables singly, and together, on mathematical conceptual development. Although the model traces its theoretical foundations to the psychological literature on conceptual development, this in and of itself does not guarantee its effectiveness in the classroom setting.

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[1] The authors are indebted to Professor Suzanne K. Damarin of the Ohio State University for her valuable suggestions on the issue.

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TOWARDS A MODEL FOR THE TEACHING OF MATHEMATICS AS GOAL-DIRECTED ACTIVITY

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The aim of this paper is to describe some preliminary work towards the construction of a model of mathematics learning and teaching which can be useful to the elementary school teacher. Progress so far has resulted in what might be termed a "pre-model": an isolation and simplification of certain aspects of the learning and teaching process and the fitting together of these parts in a certain pattern. To avoid awkwardness, this "pre-model" will be called a model in this paper. But the reader should remember that the term refers to an embryonic, rather than a finished, model. Both the theoretical perspective of the model and the model itself can aid in understanding specific classroom behaviors, in selecting and designing educational experiences, and in comparing different types of mathematics programs. The model and the theoretical perspective from which it comes will first be described. Examples from the classroom will be given in this section so that the model's application to the mathematics classroom will be clear. The second section will discuss some changes with respect to particular steps of the model which are reflected in the learner over time. The third section will discuss educational implications of the model more generally.

A Model of Teaching Mathematics as Goal-Directed Activity

This model assumes that mathematics learning is an action, an operation, an activity-- something which one does. This focus is particularly appropriate for mathematics, for so much of mathematics learning concerns operations on mathematical objects. This focus is also particularly appropriate for young children, for a primary mode of learning for them is through action on objects (e.g. Piaget, 1965). An assumption of the model is that the aim of mathematics instruction should not be merely the ability to do mathematical activities or operations isolated from some context or only when assigned by a teacher, but that these activities or operations be experienced by the child as set within a total goal-directed action sequence. That is, the child should be able to encounter a mathematical situation, set a sensible goal, think of a means to reach this goal, accomplish this means, and relate the accomplished means to the goal. Additional assumptions of the model are that the young child is initially unable to complete many such goal-directed activity sequences by himself and that it is the task of the teacher to help the child learn to do so.

The model for goal-directed activity in the classroom appears in Figure 1. It consists of a simple outline of goal-directed activity for both the teacher and the child and of relationships between these outlines. The possible relationships are several and complex. For the discussion that follows, it will be useful if one considers each entry in the flowcharts as being potentially rather than actually carried out. That is, at any given moment only some of the entries for either the teacher or the child will have been carried out, and sometimes only the teacher's or only the child's entries will be involved.

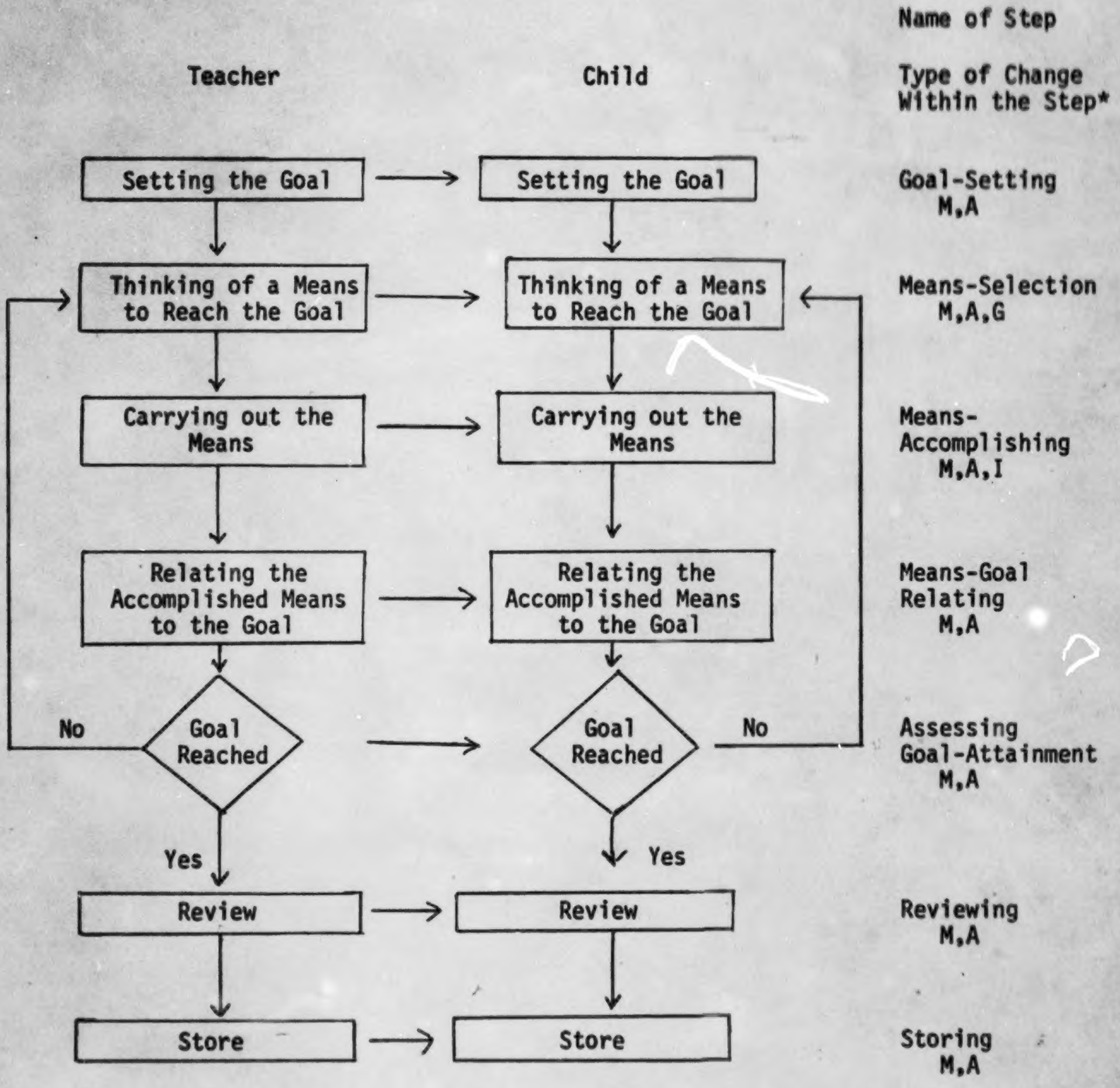
For a new mathematical activity, the teacher is capable of doing the whole goal-directed sequence but the child is not. The horizontal arrows between an entry in the teacher's flowchart and the corresponding one in the child's indicates that a teacher is helping a child to do that entry. For a given activity, a given child may be capable of doing none, one, or several of the entries in his flowchart. The teacher must ascertain which the child is able to do and help him to do the remaining ones. The aim is to enable the child to complete the whole sequence unaided by the teacher. Thus, the paradigmatic case is of learning which begins with the teacher doing all of the entries and the child none, then gradually moves to the child taking over some of the entries while the teacher continues to do some, and ends with the child doing the whole sequence independently. In that interim period, the flowchart of the actual sequence would have some entries done entirely by the teacher, some entirely by the child, and some done jointly. Of course a child may not always replicate a particular step of a teacher; rather a correct or an incorrect alternative step may be substituted by the child for one or more steps.

The theoretical perspective of this model is a Soviet one, primarily that of Vygotsky (1962, 1978) and of Leontiev (1978). Vygotsky hypothesized that children learn goal-directed behavior as they learn many other things, through a gradual development from the inter-psychological plane to the intra-psychological plane. That is, children become capable of accomplishing certain goal-directed activities through their interactions with someone who knows how to do that activity (inter-psychological involvement), and they gradually build the knowledge gained from this interaction in their own heads (the intra-psychological plane). The paradigmatic case described above is the interpretation in the model of this inter- to intra-psychological movement. It can occur with one learner at a time or with a group or a whole class. This movement may involve a kind of fading process from specific directions for each step to direct questions and then to more subtle hints. With practice, the learners become able both to do and to remember the whole sequence. Whenever more than one learner is involved in the same fading process, problems of the fit of the fading process may arise. Any particular pace will slow down some students and be too fast for others. Sensitivity and knowledge of individual learners is required by the teacher for a fading process or processes that will work reasonably well for most of the students.

A relatively trivial application of the model concerns the child doing a mathematics workbook page. The child sets the goal (or the teacher sets the goal and the child acquiesces) of "finishing the workbook page." He then looks at the page to see whether one of his old procedures will do for this page or whether he will have to figure out something new. If a new procedure is required, he may be able to figure it out by following the instructions given, or he may have to ask the teacher or a friend. In any case, his question is not "what am I learning about here?" but "what am I supposed to do here?" Once a means is chosen, it is carried out correctly (or incorrectly) and the goal is achieved (or not achieved).

Figure 1

Goal-Directed Activity in the Classroom


Name of Step
Type of Change Within the Step*
Teacher
Child
Setting the Goal
Setting the Goal
Goal-Setting
M,A
Thinking of a Means to Reach the Goal
Thinking of a Means to Reach the Goal
Means-Selection
M,A,G
Carrying out the Means
Carrying out the Means
Means-Accomplishing
M,A,I
Relating the Accomplished Means to the Goal
Relating the Accomplished Means to the Goal
Means-Goal Relating
M,A
No
Goal Reached
Goal Reached
No
Assessing Goal-Attainment
M,A
Yes
Yes
Review
Review
Reviewing
M,A
Store
Store
Storing
M,A

*M: Mastery
A: Abbreviation

G: Generalization
I: Internalization

The behaviors that gradually become a part of goal-directed activity can be thought of as isolated imitative action sequences. These action sequences are learned from parents, older children, television, and other sources as well as in school. They can be thought of as little procedures which children learn and apply with varying degrees of attached meaning. Some of these behaviors or sequences of behavior are important ones and are used in the construction of other mathematical ideas (e.g., counting objects). Others are trivial and/or serve only as an end in themselves. Many of the latter sort of procedure appear in mathematics workbooks. There is the "draw lines between the objects and say the one with left over objects" procedure, the "write the numbers in the first sentence (3 + 6 = 9) backwards in the second sentence (_ - _ = _)" procedure, and the "put the alligator jaws (>) in the blank so it eats the bigger number" procedure. Many of these procedures have more to do with mechanics of the practice lay-out or symbols than they do with the concept involved. For example, children may have difficulties orienting the > symbol correctly while knowing quite well which number is larger. Practice of some procedural skills is important. But the more difficult and important tasks for mathematics instruction are to ensure that all of this "doing" has some meaning attached to it. Setting these isolated imitative action sequences within a realistic goal-directed activity is one way to accomplish this. Assessing the actual goals of each assignment is another.

This model indicates some problem areas in mathematics learning. These can occur at each step of the model. The goal of the teacher or text or workbook author may not be the goal adopted by the child, and thus the teacher and the child may be carrying out different action sequences. The teacher's goal for a ditto sheet may be "the child learns about measuring with centimeters" while the child's goal may be "to fill in all of the blanks correctly." If that is his goal, the child is being very clever (i.e., really choosing the best means of reaching that goal) if he copies the answers from the teacher's copy of the ditto sheet. His behavior is quite sensible when viewed in the light of *his* goal. Behavior exhibited by some children during mathematics learning may seem purposeless to a teacher. It probably is not. It is just purposeful with respect to the child's goal rather than to the teacher's goal.

A child may set the same goal that the teacher intended ($G_{teacher} = G_{child}$), but may select a different means ($M_{teacher} \neq M_{child}$). A thoughtless (or insecure or harried) teacher will conclude automatically that a non-expected choice of means is an error and will impose her selected means upon the child. This judgment may be incorrect. The unexpected means chosen by the child may, in fact, be a perfectly adequate one for reaching the goal. Even if it is not, the teacher can later help the child relate his means to the goal and begin to understand why an alternative means would have been better. For both of these reasons, a knowledgeable and sensitive teacher will sometimes let a child carry out his alternative plan and then will later help the child in reviewing the adequacy of his plan. In this way, children can begin to learn how to evaluate and correct their own activities.

Once a means has been chosen, a child will proceed to carry out this means (Means-Accomplishing). This, of course, may be quite a complex step and may be done correctly or incorrectly. A later section of the paper concerning changes over time within steps will examine this Means-Accomplishing step in greater depth.

What may or may not be obvious is that successful completion of the first three steps does not guarantee that the next step will be successfully executed, i.e., a child may do the Means-Accomplishment step and still not be able to relate the result of the accomplished means to the goal or to the original problem situation. For example, a child may apply counting in situations in which it is appropriate to do so and still not be able to use the results of the counting activity. One of the tasks in Saxe (1977) is for children to use objects to make a model copy of a standard group of objects. Some children make an approximate copy without counting. If counting is suggested, they count the standard set and then the copy, but are not able to use this counting information to improve their answer. They are unable to relate their accomplished means to their original goal, and they continue to rely simply on their perceptual evaluation of the situation. A similar phenomenon occurs in Piagetian tasks for conservation of number. Some children will count the more distributed set and say "That has seven," and then count the more compact set and say "That has seven," and then still state that the spread out set has more (e.g., Green & Laxon, 1970). Working story problems provides another example of a failure to accomplish the fourth step. Some children can determine by contextual cues which kind of "game" they are supposed to play for that problem ("Oh, I need to do take away"), but they are unable to relate their subtraction answer to the original problem. However, with many other types of means and goals the fourth step is a relatively trivial one. Thus the Means-Goal Relating step may be negotiated with ease or with difficulty depending upon the particular means and situation involved as well as upon the ability of the child using the means.

The fifth step, Assessing Goal-Attainment, is closely related to the previous step (relating the accomplished means to the goal). If the fourth step has been accomplished successfully and if the means was carried out correctly, assessing whether the goal was reached is fairly simple. If the goal involved simply "getting some answer," only the most outrageous answer will probably be noticed and the answer to the decision of goal attainment is quite likely to be "yes." If the goal instead was "getting a sensible answer," the output from the means will be assessed a bit more carefully with respect to the original goal. In this case, a judgment that the goal was not reached might be made for two reasons. First, the means might have been carried out incorrectly, and it might need to be repeated. Second, the means might now be seen to be inadequate for reaching the goal, and a new means might be chosen. One of the aims of mathematics instruction is to lead children to set the latter kind of goal so that this step of Assessing Goal-Attainment is a more meaningful one.

The sixth step, Reviewing, is a fairly difficult one. This step is a consolidation step in goal-directed activity in which one quickly reviews the whole goal-directed sequence, re-relates the means used to the goal, recalls anything special that occurred, and perhaps evaluates the effectiveness of the particular means used. This step is not necessarily a long analysis but is rather a quick review (re-view) of the just completed activity. This step more than any other in the sequence is one that requires the help of the teacher. Most young children do not have the long-term strategic orientation that will lead them to do this reviewing spontaneously. The teacher's questions can lead them through this process, and the process can gradually become one which they will do themselves. The same questions that are used for the first five steps can be used here again (e.g., "Now what were you trying to do here?," "What did you actually do?," etc.), but this second talking process can begin to bring together the whole sequence in a way not possible when the child is involved in each step initially. This step involves an attempt by the teacher to bring together all the trees (steps of the process) into a single viewable forest.

The last step (Storing) acknowledges the purpose of the previous step--to store the reviewed sequence for use at a later time. We at this time know very little about how this storage takes place, about what will facilitate it, or about its effects. Effective storage processes may result in more competent behavior in particular situations and may positively affect transfer to related situations.

Types of Learner Change Across Time

Now that the static model has been described, some ways in which change occurs within certain steps of the model will be presented. These changes affect the amount of teacher intervention required for particular steps of the model. These changes are adapted from those described by Gal'perin (1957). Gal'perin discussed four independent types of change: mastery (familiarity), completeness (abbreviation), generalization, and assimilation (level of internalization). We shall call these types of change mastery, abbreviation, generalization, and internalization, respectively. The following discussion of these types of change, though based roughly on Gal'perin, represents modifications, clarification, and extensions of Gal'perin's ideas. In particular, the application of these types of change to the model of goal-directed activity are entirely those of the present author.

Two of the types of change, mastery and abbreviation, apply to all of the steps of the model, while generalization applies to the Means-Selection step and internalization applies to the Means-Accomplishment step. Each of these types of change will be discussed in turn.

Mastery. Mastery is determined by the speed and accuracy with which a step in the goal-directed sequence is done. Performance on each step can move from an initial low mastery to a higher level of mastery. The level of mastery for many of the steps will vary with the particular goal set and activity done. Mastery in such steps as Goal-Setting and Reviewing may be more task independent than in other steps, and mastery seems to have particular implications for the Means-Selection and the Means-Accomplishing steps.

Mastery of Means-Selection with a particular means would seem to increase as a child uses that means. What may not be quite so obvious is that the ability to select a particular means for a goal is affected by the ability to recall that means at the appropriate time. There is a very large body of psychological research on recall memory (e.g.,Kail & Hagen, 1977). In general, increased familiarity increases the rate of recall. Thus increased experience with a means seems likely to increase its rate of recall. Thus the familiar phenomenon of a child selecting a less efficient means of reaching a goal because it is more familiar to him may sometimes be a result of the fact that the less efficient means is more frequently recalled earlier than the more efficient but less familiar means. Because the Means-Selection step for young children is more like "finding any means that will work" than "recalling all possible means and carefully selecting the best one," the means that is recalled first is likely to be the one that will be used.

Sometimes the choice of a particular means becomes so automatized that a new means has little chance of getting used. In this case, paying particular attention to the Means-Selection step might enable a child to deautomatize that means and begin to substitute a new, more efficient means. That this substitution process is a difficult one is the assumption underlying arguments against teaching children some helpful middle steps (sometimes labeled "crutches") in certain processes. Conscious attention by the child and the teacher to the Means-Selection step when transition to a new process begins might enable a child both to benefit from the helpful extra step when he needs it and to wean himself from it when appropriate. This latter weaning step also depends on the level of mastery of the new process. If the new process is really significantly more difficult than the old one, the level of mastery for the new may never equal that of the old. In that case, it is at least an open question as to which process the child really ought to use (in spite of the seemingly obvious "superiority" of the new process).

Many factors can influence the speed with which a given child increases his level of mastery of a particular means. One of the most powerful of these factors is the level of mastery of subskills involved in that particular means. Ascertaining the subskills involved in a particular means requires some sort of task analysis of that means. This analysis may be a mathematical one, a common sense "intuitive" one, or an empirical one. Most such analyses are, of course, some mixture of these, even if one component is relatively very small. There are several research studies in the area of early number activities that indicate that this task analysis approach can be an effective one in determining developmental relationships between particular abilities. These studies are reviewed in Fuson (in press), where they are also discussed in the context of Vygotsky's "zone of proximal development." Studies which examine the effect of subskills on teaching a complex skill (e.g., Brownell, 1953) are also valuable in helping us understand what may contribute to the mastery of a particular means.

Recall of steps and sequences of steps is also important in Mastery. Drill and practice activities which practice these steps operate under this assumption. But most of this drill is designed on an ad hoc basis, and little of the psychological, theoretical, and empirical literature on recall memory has been utilized in designing mathematics activities for children.

Mastery is an important part of mathematics learning. However, it is important to remember that mastery of isolated activity sequences in artificial situations does not ensure the application of these action sequences at the right time or place. The latter can only come about if children have practice in applying these action sequences in non-trivial, meaningful situations, i.e., in situations in which they can exhibit meaningful goal-directed activity. The theoretical and empirical work of William A. Brownell (Weaver & Kilpatrick, 1972 a,b) on the relationship between meaning and mathematics learning is an example of attempts to extend the concept of mastery beyond its most limited sense.

Abbreviation. Abbreviation or completeness of an activity refers to the changes which occur in the following progression: from the initial, laborious focus on each tiny step of an activity to a well-integrated single activity sequence to an automatized activity sequence to a final sequence in which steps or sequences of steps are omitted as they themselves become automatized. One example of Abbreviation is that of a child performing the addition algorithm. Initially there may be a talking-through to oneself of the whole process (a scene observed at my breakfast table recently): "Six and six are twelve. Write down the two and carry the one. One and seven and five are ... (pause representing an abbreviated step) thirteen. Write down the three and carry the one. One and three and four make eight. Write down the eight. Gee, addition is easy." As the addition facts themselves become more automatic, the talking through and the recording of the carry will gradually drop out of the process. Another simple example of Abbreviation is the change from counting all (i.e., for five objects plus two objects, counting the five, then the two, then all seven objects) to counting on (counting the five objects and then counting on two more--six, seven).

Though abbreviation of mathematical activities to the automatic level is one of the chief goals of school mathematics, we still know little about the processes that are involved in or lead to this automaticity. Schools try to automatize various Means-Accomplishing skills by stimulus-response types of practice of specific steps in a process (e.g., flashcards) and by practice of a whole process (e.g., pages of problems) but pay very little attention to helping children help themselves through the whole process in an initial non-abbreviated manner. Even less attention is paid toward abbreviation at any of the other steps of the goal-directed sequence.

Generalization. Generalization is extremely important in mathematics learning because of the way in which this learning occurs. Rather than learning an operation (e.g., addition) in general, a child begins to learn about it by doing it with particular numbers. He must generalize the operation itself and also generalize in some fashion the situations in which it is appropriate to do this activity, i.e., the goals for which this activity will be the appropriate means. These goals to which an activity is generalizable may differ from the original in their perceptual attributes, in their conceptual complexity, in their mathematical difficulty, and in their situational context. Though there are some research studies and a few theoretical notions (see, for example, Post & Reys and Brazier & Mick, this volume) which bear on the question of generalizability of action sequences, at present this area is poorly conceptualized and under-researched. Because it is such an important area for mathematics learning, it deserves more direct attention than it has had up to the present. Because generalizability is so important, a few research studies from which principles concerning generalizability can be drawn will be discussed briefly.

Some studies on the development of early number concepts do provide information about some types of generalizability of mathematical activities. The size of the number involved in a counting activity is the single most important influence on the generalizability of a large number of tasks which involve counting. Children can count out a set of one or two items as young as age 2-1/2, but do not consistently do so accurately for five items until about 5-1/2 (Gelman, 1972b). Forty-two percent of a group of first grade children could on three trials correctly count a set of ten objects thrown on the table, while 63% could do so for five objects (Buckingham & McLatchy, 1930). Gelman's research with very young children (1972 a,b; Gelman & Tucker, 1975) indicates that children as young as 2-1/2 successfully use counting to solve conservation tasks for sets of two and three objects, but not for larger sets. Children appear to be able to apply a cardinality rule ("Count the objects and assign the last number said as the number of objects") to sets of one to four chips before they can successfully do so for sets of five to seven chips (Schaeffer, Eggleston, & Scott, 1974). Thus, the activity of counting may be able to be used for a variety of goals--getting a certain amount of objects, finding out how many objects someone else has, deciding if there are still as many objects as there were before--for the numbers one and two first, then for the numbers three and four, and still later for five through nine.

One might expect the generalization process with respect to activities with numbers to continue fairly regularly as numbers get larger and finally to generalize to the getting larger process itself, i.e., to some level of understanding of an activity as applied with any number at all. This probably describes fairly accurately what does happen with respect to the string of number names, i.e., to the words used in counting. But the generalization process for symbols is much more complex. As soon as symbols for numbers above nine are introduced, all of the complexities of the base system interact with the fact that the numbers also are simply getting larger. Our number symbols and our number names are different mathematically. These differences and their implications for mathematics teaching are described in more detail in Bell, Fuson, and Lesh (1976, pp. 421, 429-436). These differences between the systems of number names and of number symbols and, in particular, the problems which are raised by generalization of old activities to the mathematically more complex system of symbols have received too little attention by researchers. An exception is the work of Dale Underwood of Governors State University (Note 1, Note 2). He has been focusing on just this area of transition from number activities which use the string of number names to those activities which use grouping into tens. He has found that the ability of children to know that transformations of various sets of objects leaves the numerosity of the set invariant (i.e., conservation of number) occurs on at least three levels of generalization. These levels he calls conservation of counting (which is a combination of conservation of numerosity and of the point counting procedure as applied to one set), conservation of correspondence (the traditional Piagetian two set conservation task), and conservation of grouped number (the invariance of a set of objects regardless of whether it consists of all ones or of groups of tens and ones). Additional conceptual or empirical research is needed to focus on this difficult area of generalization caused by the different levels of mathematical complexity of the system of number names and the system of number symbols.

Zoltan Dienes (1960, 1963, 1964, 1966 a,b,c,d,e) has invented physical embodiments and written curriculum materials which enable many complex mathematical concepts to be learned through directed activity, i.e., through acting on objects in a goal-directed manner. He has articulated two theoretical principles which relate to the generalizability of mathematical activities. These are the mathematical variability principle and the perceptual variability (or multiple embodiment) principle. The first stresses that if the goal is to learn about a concept that involves some mathematical variable, experiences need to be provided with many different exemplars of that given variable. For example, if children are learning about the base systems of numeration, they need to have activities with several different systems. The perceptual variability principle proposes that a child do activities with materials which possess the same underlying mathematical structure but are perceptually different. Thus, in learning to count objects children should have an opportunity to count a wide range of different objects, including sets of identical objects as well as sets of objects differing in various ways. Systematic research on the timing of and constraints on characteristics of different embodiments is required before these principles will be able to guide in any detailed way the construction of curriculum materials, although some teachers and some materials (e.g., Montessori materials) have independently come to use similar principles in a general way.

Internalization. Internalization applies only to the Means-Accomplishing step because it is the only step which is, or can be, carried out initially in the external world and must eventually be internalized. All of the other steps are, or can be, internal ones from the beginning, though overt verbalization about any of the steps is possible and may occur at certain times. Gal'perin indicated that internalization was the crucial process in mathematics learning. He described five steps, or levels, in the internalization process and asserted that mathematics learning which did not go through these five steps was formal, empty, essentially non-learning. His five levels are:

1. Creating a preliminary conception of the task (i.e., the means)
2. Mastering the action using objects
3. Mastering the action on the plane of audible speech

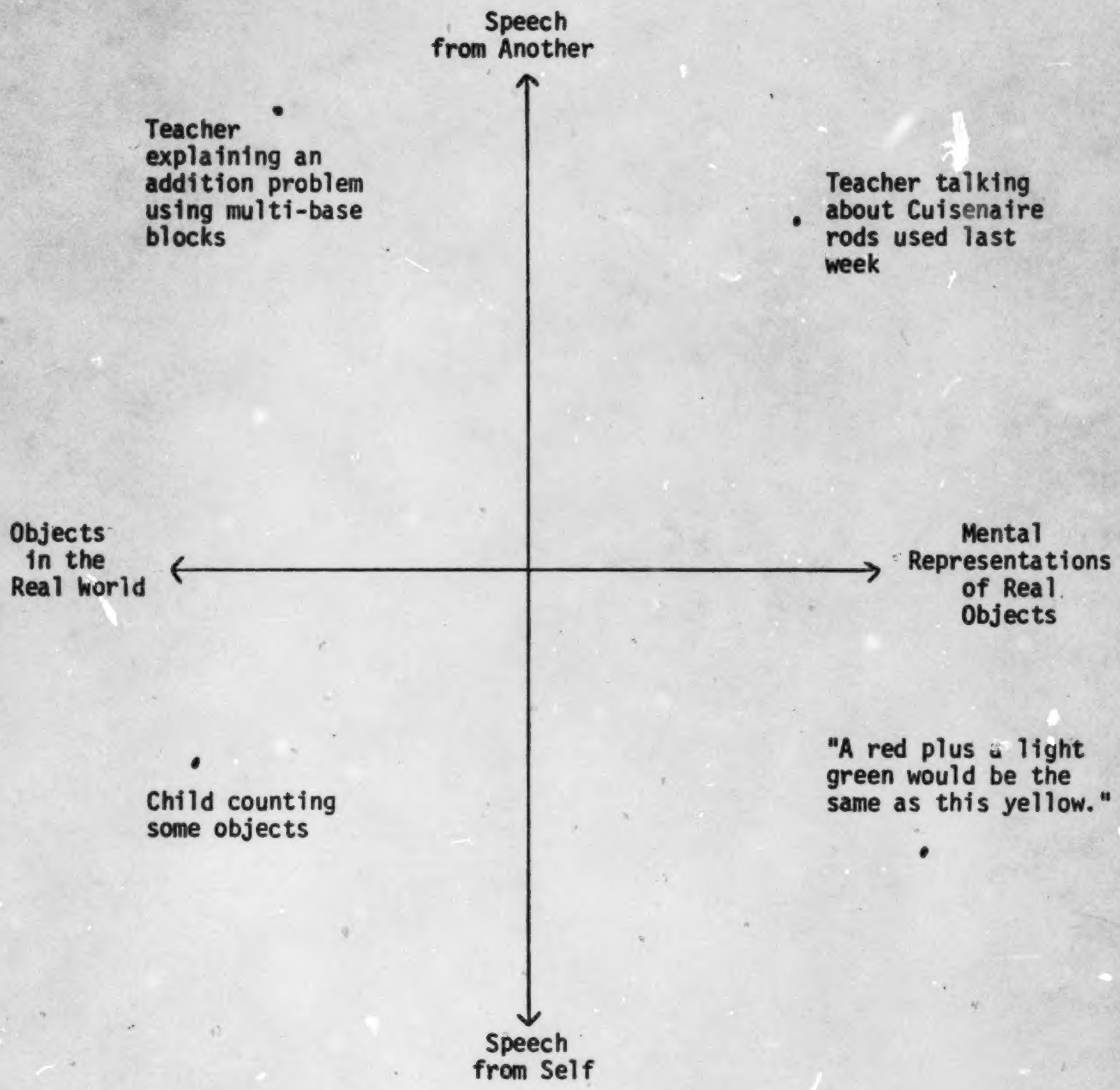
4. Transferring the action to the mental plane
5. Consolidating the mental action

Steps 2 and 4 look very Piagetian--one reproduces mentally the actions made in reality. Step 3 contains the usual Soviet focus on the self-regulating use of speech. Step 1 is a step that probably is often ignored in teaching, with the result that the child is not able to begin accomplishing the means because of a state of confusion. Step 5 is a necessary but very fuzzy step that probably is more related to Mastery or to Abbreviation than to Internalization *per se*. We will therefore omit step 5 in the following discussion of Internalization.

Steps 2, 3, and 4 focus on important aspects of internalization. But we would like to suggest that they indicate two independent types rather than one type of internalization. These two types can be thought of as two orthogonal axes representing one continuum from use of concrete objects to the use of internal representations of external objects and another continuum from relevant speech from another to speech from or within self. Each of the quadrants defined by these axes contain particular types of the mathematics teaching-learning process. Some examples of these are indicated in Figure 2. Movement along the horizontal axis (from objects to representations of objects) has often been described and has intuitively been used by teachers. Downward movement along the vertical axis represents the hypothesized movement from the inter-psychological to the intra-psychological plane. In the model proposed in Figure 1, this type of movement could occur at each of the steps. This movement indicates a shift from teacher directives to increasingly indirect questions to self-questions by the child. Such a movement also encompasses self-regulating speech which may develop as the teacher describes each step in some process and the student later says the same words to himself to help himself through the process.

The development of self-regulating speech has been the focus of a considerable amount of psychological literature in the past twelve years. This literature is reviewed in Fuson (in press). Only one study has been reported involving the use of such self-regulating speech in mathematics learning. Pechman (1978) recorded the amount of such speech children aged 6-1/2 to 10-1/2 years used while they did teacher-assigned tasks in the elementary classroom (some of these were mathematics assignments). During the early transitional (Piagetian) operational phase, use of self-regulating speech or muttering correlated +.87 with standardized scores on reading achievement and +.62 with scores on mathematics achievement. Pechman noted that children seemed to use overt self-regulating speech when tasks got more difficult or harder to understand. She described one example from mathematics in which a child silently mouthed almost all the numbers on her math paper which were under twenty but consistently voiced loudly to herself those over twenty (personal communication). These Pechman results indicate that teachers should allow children to use such facilitative speech-for-self. They suggest that we might also examine the potential of helping children who do not spontaneously use such speech to do so.

Figure 2


Speech from Another
Teacher explaining an addition problem using multi-base blocks
Teacher talking about Cuisenaire rods used last week
Objects in the Real World
Mental Representations of Real Objects
Child counting some objects
"A red plus a light green would be the same as this yellow."
Speech from Self

Both the discussion of Gal'perin's work and of Figure 2 with respect to Internalization concern direct teacher instruction with the objects. If the media of instruction is printed matter rather than a person, an additional variable of pictorial representation is added to the internalization process. Pictorial representations are often hypothesized (by Bruner, 1973, for example) to be an intermediate stage between objects and mental representations of objects. However, the research of Sigel (1972) indicates that many children from low socio-economic backgrounds are unable to act on pictures of objects as they do on the objects themselves. In particular, children were unable to classify pictures of objects. Children's difficulties with pictorial representations, and with the even more abstract drawing and schematic representations common in mathematics books, have been relatively little researched. Such representations may slow down, rather than speed up, the internalization process. Because the relationship of pictorial representations to real objects and to mental images is not clear at this time, they are not included in the model. When such relationships become clearer, the model should be extended to include them.

Before discussing the model and its application to the classroom, some additional aspects need to be clarified. This model is simple because it is intended to describe the behavior of pre-school and early (K-3) elementary school children. Such young children only gradually acquire the ability to engage in a whole sequence of goal-directed behavior. Their behavior tends to be somewhat fragmented, with goal-directed sequences interrupted by competing stimuli. The ability to spend a major proportion of the mathematics learning time in productive goal-directed activity would be an important and major achievement for third graders. For older children, the model can be extended by adding such steps as planning (i.e., mentally performing and evaluating various alternatives before they are done). However, because humans frequently function at less than their maximal ability, the present model may describe much of the mathematical behavior of fourth through seventh graders and a significant amount of that of older students. Thus, the present model can be considered as either a precursor model which requires detailed steps to be added to it as the actors described by it become capable of more complicated behavior, or as a model of a simpler kind of behavior in which all people sometimes engage, though they may be capable of behaving in a more complex fashion when necessary.

The model is simple for another reason. This model is intended for use by teachers while they are teaching. Thus it must be short and simple enough to be carried in the head. A longer, more complex model might be more useful for a detailed theoretical or empirical analysis of behavior, but such a model would be too difficult to remember and the scale of analysis too fine for a teacher who must not only reflect upon her actions but also must act.

A final note concerning the relationship of this model to models of problem-solving is in order here. One difference between this model and problem-solving models is that the latter usually are designed to apply to situations in which a "problem" exists, i.e., in which there is a gap between some goal a person seeks to attain and his immediate attainment of that goal. The present model of goal-directed activity applies to these "problem" situations, but in addition applies to those situations in which there is no such gap or the gap is a very small one. Another difference is that the problem-solving models from psychology are models of adult processing and are rarely applied to mathematics problems. The problem-solving models in mathematics education also are intended for older children (fourth graders and older) or adults. Thus both of these types of models have characteristics which limit their applicability to young children. They may require cognitive processes not available to young children, they may not be specific enough to apply to mathematical situations, and they may be so specific and detailed that they require more mathematics than the young child knows. This model and the role described for the teacher are very Polya-like (Polya, 1956). They are different from those described by Polya in the above ways, however, and the Polya-steps involve a much more sophisticated and complex behavior than do those in the present model. For example, Polya's first step is "Understanding the problem." Our first step, "Setting the goal," can be quite simple and does not even require that the problem be understood in order for some sort of goal to be set.

Implications for Instruction

The model can be used for diagnostic purposes. If a child is having difficulty in some mathematical situation, it may not be immediately obvious which of the steps is the source of the difficulty. The goal set by the child may be incompatible with desirable mathematical outcomes in that particular situation (this may be a cognitive or a motivational difficulty) or the child may not have been able to set any goal. The child may have understood and accepted some task demand but not be able to determine how to accomplish this task. The child may be having difficulty with some part of the execution of the means, e.g., he may have decided that he is supposed to multiply fractions but not be able to remember how to do so. Or, the means may have been carried out but the result not connected to the problem-situation. A series of teacher questions ("What are you trying to do here?", "How are you trying to do it?", "Let's look at just what you are doing.") can help both the teacher and the child discover at which step the source of the difficulty lies. This series of questions might eventually be posed by the child to himself in attempting to make himself more aware of the steps in his goal-directed activity.

In addition to this diagnostic use, the model can be used to plan classroom activities for children. First, specific steps of the model can be isolated and focused upon. Teachers could focus on Goal Setting by talking with a group of children about goals different children have set, about why they might have set these goals, and about alternative goals some of them might have set. Another technique would be to formalize the goal setting process in some way, such as by performance contracts. Such contracts can be a useful means to aid students in attending to the whole goal-directed cycle, from setting the goal to relating the accomplished means to the original goal. Contracts do run the risk of becoming only mere agreements about how many workbook pages a child will try to fill out. Even in this limited way, however, contracts may help to focus the child more on the whole goal-directed cycle than does constant teacher assignment of pages. The latter offers the child only two choices: to concur or not to concur with this choice. In this case, the goal setting process by the child is not likely to be a very deep or involving one.

Thinking-of-a-Means can be focused on specifically by activities in which different means will accomplish the goal. The Nuffield Project books (1967 to 1971) and USMES (1970-1975) are a rich source of such open-ended problems. Such open situations are in sharp contrast to usual classroom practices. Teacher and textbook activities are too often set up with one intended specific means to achieve the goal. Children are creative enough (and sometimes obtuse enough) that this single means approach may be thwarted. But even then, a child may be accused of using the "wrong" method. Through experiences in which alternative means are possible and through comparisons of different means which may occur in the Review and Store steps, the Means Selection step may gradually move from being "using the first means that pops into your head" to "carefully selecting the best means from considered alternatives." Of course, for familiar goals, an automatic selection of the means is desirable (and would be an Abbreviation). But for new situations, a considered Means-Selection is probably desirable.

Very short but frequent classroom focusing on the final Review would be very helpful to students. A short discussion with an individual or with a class at the end of a classroom period can review what was done and provide a sense of closure for students who may tend to "lose sight of the forest for the trees" while they are working through some activity. A summing-up by the teacher and students together can help to alleviate the state that leads to children giving to the question "What did you do today?" typical indefinite responses such as "I don't know," or "I don't remember." This summing-up is also then available and can be referred to by the teacher or children the next time a similar goal situation is met ("Remember when...").

Much of the activity in the ordinary mathematics classroom is focused on the Accomplishing-Means step. The teacher or the textbook effectively withdraws the goal-setting and the means selection steps from the participation process by determining them for the child; the child then is to concentrate on perfecting his skill in carrying out that particular means correctly. This is a necessary and effective strategy for helping children focus on mastering specific means. But once this is accomplished, the processes perfected in the Accomplishing Means step need to be placed into the whole cycle of goal-directed activity so that the child gets practice in determining in which situation he should use them. This is true of all of the steps. They can be concentrated on individually, but then it is essential that they be followed by experiences which enable a child to go through all the steps of goal-directed activity.

In addition to suggesting certain classroom practices, the model elucidates relationships between certain types of classroom organizations. Traditional teacher-centered classrooms place very little emphasis on alternatives at the Goal-Selection or Means-Selection steps and tend to spend a considerable amount of time on drill and practice of Means-Accomplishment steps. They either neglect the first two steps of Internalization entirely or move through the process much too rapidly. Open classrooms try to present choices at the Goal-Selection step and give some open-ended problems with several possible means available at the Means-Selection step. But some do not provide sufficient guidance to accompany the choices that are offered, and thus too little of some children's time is spent in productive goal-directed mathematical activity. Some of these open programs also do not place enough emphasis on the Means-Accomplishing step to enable children to achieve Mastery in very many areas.

Specific types of school mathematics programs also tend to emphasize certain aspects of the model to the exclusion of others. Programs which stress "back to basics" stress Abbreviation and Mastery. "New math" programs are concerned with Generalization, as one of the chief foci of this movement in the 1960's was the search for basic mathematical ideas which would generalize; these powerful generalizable ideas then were proposed as the foundation of the proposed mathematics programs. The movement toward using manipulative materials in mathematics classrooms, the research and theories of Jean Piaget, and the writing and embodiments of Zoltan Dienes all have been concerned with portions of the Internalization process.

Each of these types of mathematics programs contributes something important to the learning of mathematics. The model indicates how these contributions may relate to each other. It may be that to move children to as high a level as they are capable of reaching on all four types of change requires more time than we are presently willing to give to school mathematics learning. If this turns out to be true, then the model provides a useful general framework from which to make choices of priorities. Even if enough time can be provided and such difficult choices do not have to be made, more knowledge than we presently possess about development and about learning is required to maximize children's learning. Some of these areas of required research with respect to young children's mathematical activity have been indicated in this paper.

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2. Underwood, D. Personal communication, April 22, 1977.

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PRELIMINARY STEPS TOWARD A COGNITIVE MODEL OF LEARNING PRIMARY MATHEMATICS

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This paper presents a survey of a group of research projects that are in progress. The general goal of this research program is to contribute to a more adequate understanding of the learning of elementary addition and subtraction in the primary grades. Criteria for such a theory are that the psychological processes involved in learning should be specified in ways that include the specific content that is learned in arithmetic, and that the theory should be developed as clearly and with as great a degree of formal precision as possible.

A cognitive theory of learning should have three components (cf. Glaser, 1976). First, there should be a theory of the knowledge that is acquired as a result of learning. This model of the learning outcome is constrained by but not determined by the mathematical structure of the material that is learned. The outcome model describes cognitive structures and processes that enable a student to perform the tasks that are used to test whether the student has learned successfully. Second, there should be a theory of the relevant knowledge that students have at the beginning of learning, again stated in the form of cognitive structures and procedural knowledge of students. Third, the theory should include a model of the process of transition from the initial state of knowledge to the state in which that knowledge has been acquired.

The commitment to develop psychological models of students' knowledge implies that the models will describe cognitive systems that can actually occur. When we want to characterize knowledge of some concepts or procedures, we identify tasks that depend on the knowledge in which we are interested. Then we find individuals who can perform those tasks successfully, and we use various methods to analyze the information and procedures that those individuals use when they perform the tasks. The information and procedures that we identify constitute the knowledge used to perform the tasks. We then represent the knowledge in the form of an information-processing model, and apply various experimental methods to test whether the model is an accurate representation of the processes that occur when individuals perform the tasks.

This general method depends strongly on selecting appropriate tasks to reveal the structure of knowledge, as well as finding appropriate ways to observe performance on the task to permit inferences about the knowledge

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that is used in successful performance. It is quite possible that when performance on a task has been analyzed, we may conclude that the task does not depend on the kind of knowledge we had hoped to study. Then we may choose a different task to provide the kind of information we needed, or we may change our hypotheses about the nature of the knowledge that we had considered initially.

The current preliminary state of our research on the three aspects of the general problem of developing a theory of learning primary arithmetic is described below. First, some hypotheses are presented about the nature of knowledge that is acquired by children in learning elementary addition and subtraction. Second, there is a discussion of some aspects of knowledge relevant to arithmetic that children have before they begin their formal instruction. Presented last are some tentative orienting ideas regarding the process of acquisition, involving the transition from the knowledge children have initially to the knowledge they have after about two years of instruction.

Model of Acquired Knowledge

There are two aspects of knowledge of elementary addition and subtraction about which existing research findings or research in progress supports fairly definite conclusions concerning the nature of knowledge that students acquire. These two aspects are: (a) procedures used in retrieving basic addition and subtraction facts and (b) semantic structures that are used in understanding word problems solved by use of simple addition and subtraction. Two additional aspects of knowledge can be identified, but we lack any systematic basis for hypotheses about them. These involve: (c) syntatic knowledge about the structure of arithmetic expressions and (d) relations between numerical and spatial representations of arithmetic operations.

Basic Addition and Subtraction Facts

It is customary to suppose that children simply commit the basic facts of arithmetic to memory. However, facts as such do not provide answers to questions. One wants to explain the performance of children in answering questions, and that requires a model that specifies a procedure that is used in retrieving the information needed for answering the question.

A considerable body of data supports the conclusion that children use a variety of counting procedures when they answer questions about basic addition and subtraction facts. About one-half of the preschool children (four- and five-year olds) studied by Groen and Resnick (1977) and given 20-35 15-minute sessions of practice with addition facts used the procedure shown in Figure 1. The first step is to identify the larger of the two numbers to be added. A counter is initiated at the value of the larger number and is incremented enough times to correspond to the smaller number. The answer is the final value reached in that counting process.

The procedural knowledge used for subtraction is not understood as definitely, but a fairly clear picture is available. Students typically answer a subtraction question by counting the difference between two numbers; for example, to find 7 - 5, count from five to seven and note that two units are required. However, when the gap between the numbers is large and the number to be subtracted is small, children apparently count backward to the answer (Woods, Resnick, & Groen, 1975). There is evidence that missing addend problems with the missing term in the second position, such as 5 + ? = 7, are solved in the same way as subtraction problems. When the missing term comes first, the data are unclear and we are unable to specify a simple model of the procedure or procedures that are used (Groen & Poll, 1973).

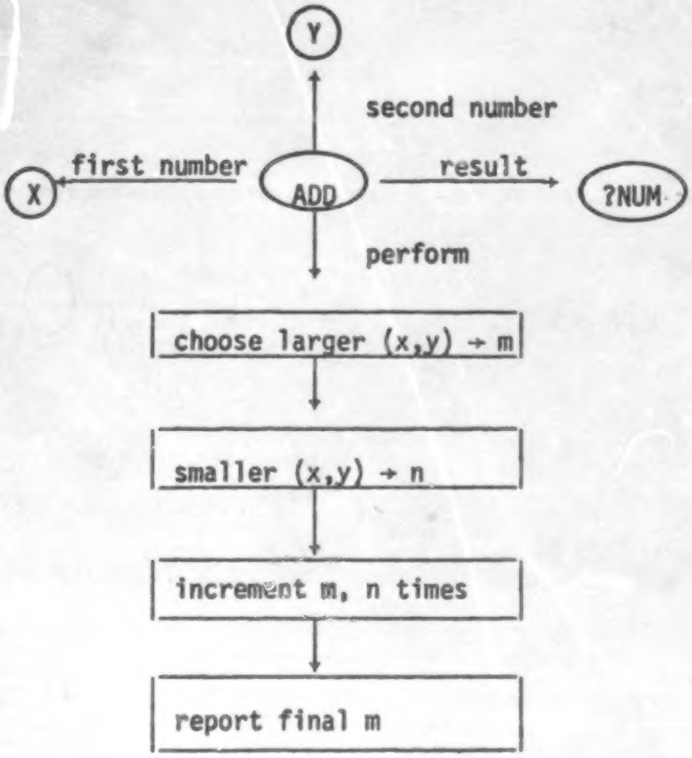

Figure 1. Procedure for answering simple addition questions.

Syntactic Knowledge

Consider the difference between 5 + 7 = ? and 5 + ? = 7. An addition procedure should be used for answering the first question, and a procedure for finding the difference should be used for the second question. To choose an appropriate procedure a student must understand the significance of the positions of the symbols in the sequence. The problem is analogous to that of distinguishing "John gave a book to David" from "John was given a book by David." The relationship between the terms depends on the form of the sentence and, therefore, understanding the relationship depends on knowledge of syntactic structure.

We do not have a good theory as yet about the syntactic knowledge that children acquire in elementary arithmetic. A suggestive result was obtained by Case (1978) in a study of missing addend problems of the form 5 + ? = 7. Case noted that most children whose performance was incorrect answered the questions by adding the two numbers given in the problem. He developed a training sequence in which diagrams were placed in the positions usually occupied by numbers in arithmetic sentences. Addition was represented as combination, where the diagram to the right of the equal sign was formed by putting the two other diagrams together. Children learned to fill in a missing diagram in the position of a missing addend, and that led to successful performance on missing addend problems. The inference that seems appropriate is that the syntactic structure of sentences involving plus and equal signs was induced from the diagrammatic representation.

Semantic Knowledge

The process of interpreting an arithmetic sentence is analogous to understanding language, since in both situations a sequence of symbols is transformed into a meaningful cognitive representation. Concepts that have been developed recently in the theory of language understanding may be useful in analyzing the understanding of arithmetic. When someone understands a sentence, the individual constructs a representation of the information in the sentence, including the concepts that are mentioned along with relevant relations among the concepts. It has become common to represent these meaning structures as graphs (e.g., Norman & Rumelhart, 1975; Schank, 1972), and Figure 2 shows a schema that might correspond to a child's understanding of simple addition sentences. For example, to understand the sentence 3 + 5 = 8, the arguments would be inserted into the structure with 3 as X, 5 as Y, and 8 as Z.

The schema in Figure 2 is not a very rich semantic representation, and it is reasonable to conjecture that this stripped-down and highly abstract representation is not one that students have when they begin to learn arithmetic. If one asks "What does 3 + 5 = 8 mean?", one may or may not get an answer; but an answer that seems fairly sensible is the following: "You start with three things, then you get five more, and that gives you eight." This semantic interpretation involves a quantity that is the starting value, a change in that quantity that involves both a direction (increase rather than decrease) and an amount, and a result of that change. These semantic relations are included in the schema shown in Figure 3. Note that in this interpretation there is implicit reference to an object that has a quantity associated with it, and a change in the quantity is the central idea involved in understanding addition in this way.

The idea of semantic meaning used here is based on the Tarskian (1959) analysis of model-theoretic semantic interpretations. Given a formal system, in this case arithmetic, a domain of objects and relations is a semantic model of the formal system if there is a mapping between the symbols and operations of the formal system into the domain such that the theorems derivable in the formal system are satisfied in the domain. The domain may contain physical objects and relations, or it may be another formal domain. For example, we say that set theory is a model of the first-order predicate calculus. In the case of arithmetic, the kind of domain that is relevant here consists of situations and events that involve quantitative relationships such as changes in the numbers of things that some persons have in their possession. It may be more comfortable for some readers to think of mathematics as such as consisting only of the formal system, and to think of a structure such as Figure 3 as referring to an application of mathematics. On the other hand, it is not unusual to consider the Tarskian interpretations of a formal system as constituting at least one aspect of the system's meaning, and I am taking that point of view in this article.

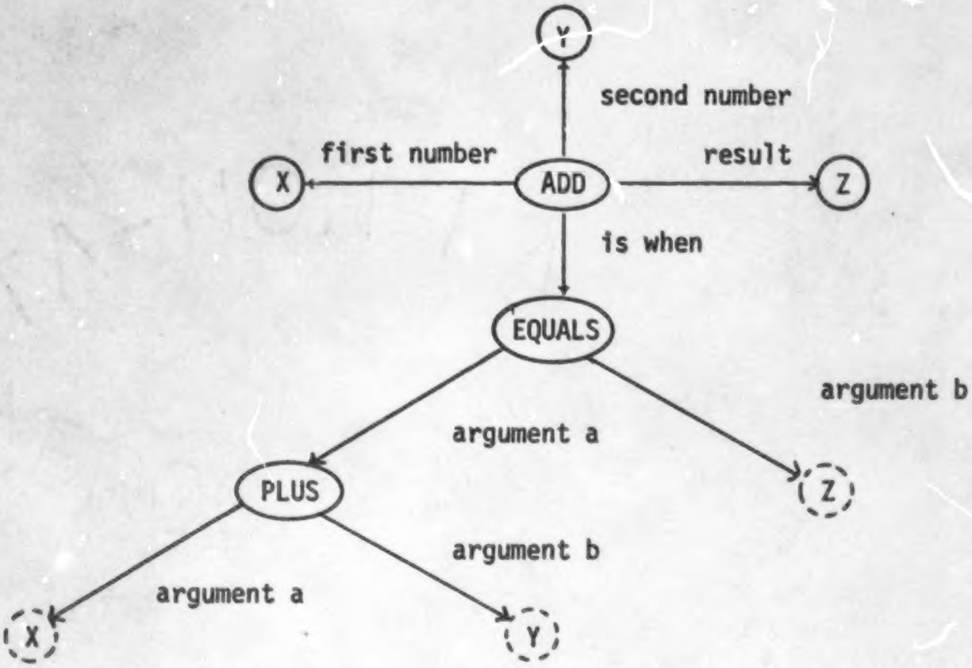

Figure 2. Schema representing relational knowledge of the concept of addition.

A schema such as Figure 3 also can be used as the basis of understanding a word problem, such as "Nancy had three eggs. She found five more eggs in the hen house. How many eggs does she have now?" This situation involves a starting quantity, an event that involves an increase in that quantity, and a new quantity that results. Therefore, the schema shown in Figure 3 provides an interpretation of the word problem, and may be the basis of a child's understanding the relationship between a word problem and the arithmetic operation that is used to solve the problem.

The ability to solve word problems is an important aspect of the knowledge of arithmetic that students acquire. An attempt is being made to develop a model of the process of solving word problems. A preliminary analysis of knowledge needed to solve problems that require addition or subtraction has been developed (Note 1).

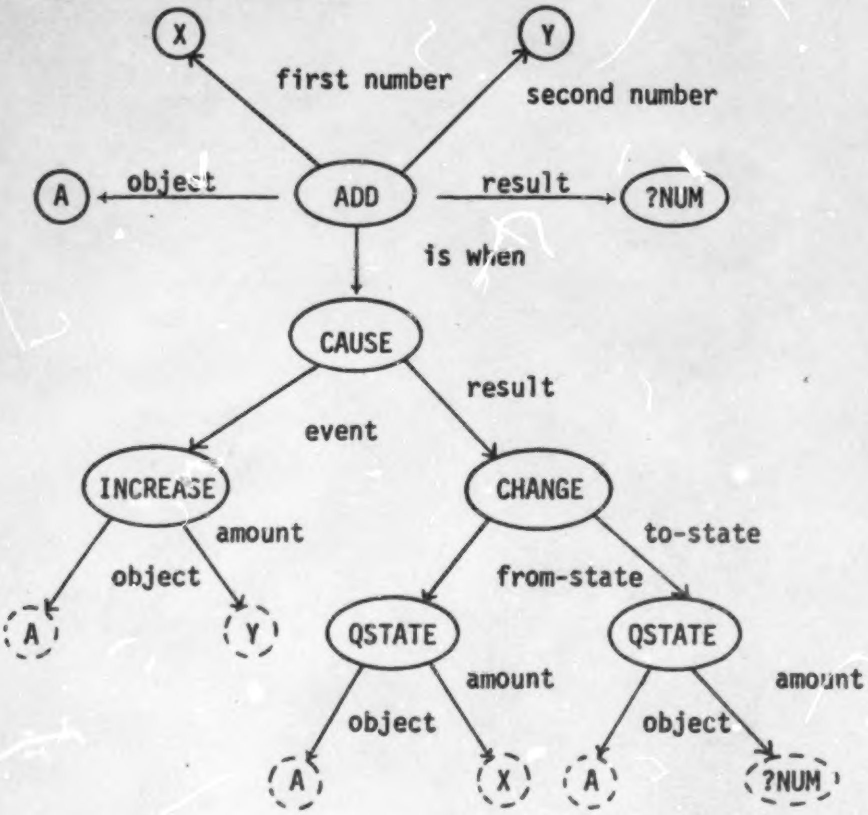
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Figure 3. Semantic schema for addition involving a change in a quantity.

The form of the model of problem solving is a computer program that takes semantically parsed representations of prose sentences as input and constructs an integrated meaning structure that is associated with an arithmetic operation. The approach represented in this model is somewhat different from that usually taken regarding these problems. The more common approach views the task as a translation from verbal text to equations. This idea was used by Bobrow (1968) in a computer program for solving verbal problems. The approach we are taking treats the task primarily as a semantic process, involving representation of appropriate relations between quantities. For the problem, "Sue had three marbles. Nancy gave her five more. How many marbles does Sue have now?" the system produces a representation in the form of Figure 3. In the representation, 3 is identified as a starting quantity, it is recognized that the situation involves increasing the quantity, and 5 is identified as the amount of the increase. The representation is generated by a production system developed by Anderson (1976) that simulates understanding by generating propositional structures from English text. The selection of an arithmetic operation--addition, in the case of this problem--is based on direct associations between semantic structures and the operators. In this example, when the representation involves a quantity and an increase of that quantity, and the question asks for the resulting quantity, the system includes the knowledge that the result is obtained by addition.

The simulation of these semantic processes required a careful examination of the variety of verbal problems that are used in arithmetic instruction. This process of task analysis has led to an interesting finding. The semantic structures that correspond to a single operation, such as addition, are not homogeneous, even at a rather general level. There are three general kinds of situations in which students must decide to add: (a) increases in some quantity, as in "Sue had three marbles, Nancy gave her five more," (b) combinations of two sets that remain distinct, as in "Sue has three marbles, Nancy has five, how many do they have altogether?" and (c) comparative specifications, as in "Sue has three marbles, and Nancy has five more marbles than Sue has. How many marbles does Nancy have?" These all require a decision to add the numbers given, but the structures that determine that decision are quite different. Our present conjecture is that children's understanding of arithmetic would be facilitated if specific instruction were given involving semantic distinctions of this kind, but a test of that conjecture has not been conducted.

There is less knowledge available than would be desirable about how primary-school children solve word problems in arithmetic, or about whether they can solve a very wide range of problems. In most classroom situations, word problems are not presented very often to first and second-grade children, probably because they may just be learning to read. However, there are some problems that children are able to solve before they have been given formal instruction in addition or subtraction. Buckingham and Maclatchy (1930) found that an average of 43.6% of simple word problems were solved correctly by children just entering the first grade. An example of the kind of problem used was, "If you have 5 oranges and you get 2 oranges more, how many oranges will you have then?" Studies of preschool and primary-grade children's ability to solve word problems are being conducted at present by investigators at Pittsburgh. Evidence collected thus far includes support for the idea that the semantic structure of a problem is an important factor in the difficulty of the problem (Note 2). We are hopeful that within the next year or two the growth of children's understanding of basic quantitative relationships relevant to addition and subtraction will be characterized in a relatively complete way.

Spatial Representations

In addition to representations related to word problems, children learn to think about arithmetic using abstract spatial systems, such as blocks, plastic counters, or the number line. These models involve physical objects, but the operations used are less concrete than the events such as finding things or buying things that are described in word problems. We have not looked systematically at abstract spatial representations in most of our research, and I am not aware of research findings that provide a basis for characterizing the ways in which children can use abstract spatial models in understanding arithmetical ideas. Perhaps a reasonable conjecture is that physical models should be considered as alternative abstract systems, rather than as concrete domains. According to this idea, children must acquire procedures for constructing and manipulating representations with physical models, just as they must acquire procedures for using numerals. However, the nature of these procedures, and the nature of knowledge needed to coordinate representations in physical and numerical domains, are questions that require future research.

Summary

The state of our work in developing a cognitive model of the knowledge that children acquire in learning arithmetic can be summarized as follows:

First, children acquire procedures for answering questions using basic arithmetic facts. We have specific models of these procedures that are reasonably well supported by experimental data.

Second, children acquire syntactic procedures for comprehending structural relations among the terms in arithmetic sentences. This knowledge is not understood as well as the procedural knowledge for answering questions, but some promising clues are available.

Third, children are able to apply their arithmetical knowledge in solving word problems. We hope to be able to develop a reasonably well formed characterization of the problem-solving knowledge that children acquire during the first and second grades on the basis of research that is currently in progress, and we have a reasonable beginning toward a model of the process of solving arithmetic word problems.

Fourth, we know that children have some capability in representing arithmetic concepts using physical objects. However, the way in which these representations are constructed, and their relationship with children's use of numerals, are not understood at present.

Model of Initial Knowledge

In parallel with the effort to develop a model of the outcome of arithmetic instruction, an attempt also is being made to represent the knowledge that children have when they enter school that is relevant to their learning of arithmetic concepts. We have been working on two issues: (a) the knowledge that preschool children have for counting objects and (b) the ability of preschool children to understand quantitative information in word problems.

Knowledge for Counting

It seems clear that children's ability to count objects is an important aspect of their knowledge relevant to their instruction in arithmetic, especially since the procedures they acquire for answering basic addition and subtraction questions appear to use counting in an integral way (Groen & Poll, 1973; Groen & Resnick, 1977; Woods, Resnick, & Groen, 1975). Thus, an appropriate model of the knowledge that children use in counting objects should be an important component of the characterization of their cognitive state at the beginning of instruction.

One view of the counting process, expressed in a theory by Klahr and Wallace (1976), is that children simply have a procedure for moving through an ordered list of the names of the integers and simultaneously tagging objects in the set of things to be counted. A modification of this simple list-plus-tagging procedure makes the system more efficient, especially at the beginning. Klahr and Wallace assumed that the number of objects in a small set can be apprehended immediately, or subitized so the counter is initialized at around four or five. However, the basic counting procedure according to the model involves simultaneously tagging objects and incrementing a list until there are no more untagged objects.

In contrast to the rather mechanistic view that Klahr and Wallace presented, Gelman and Gallistel (1978) have argued that children understand important principles such as cardinality, one-to-one correspondence, and indifference of order in which objects are counted. If Gelman and Gallistel are correct, then a model of a preschool child's knowledge of counting should be more complex than Klahr and Wallace's model in ways that reflect understanding of the principles that children understand.

The evidence that Gelman and Gallistel present for their view includes errors that young children make when they count. The fact that errors occur is not as important for theory as is the nature of the errors. This is because children's performance, even when incorrect, is systematic in some important ways and those systematic qualities reveal knowledge that the children have that is used in the task situation.

One example is Gelman and Gallistel's finding that some young children do not use the conventional sequence of number names when they count. A child may use a sequence such as, "One, two, three, six, sixteen." However, whatever sequence a child uses is used consistently, in the same order, in counting different sets of objects. Gelman and Gallistel argue that this is evidence for children's understanding of a principle of stable ordering in the sequence of number labels. When a child constructs a series of labels to use in counting, the series is structured in a way that preserves its order when it is used in different situations.

A second finding is that when children make errors in counting sets of objects, the errors do not involve mistakes in the use of labels. Errors that occur involve mistakes in the partitioning of objects. A child may count the same object twice or skip an object, but children do not use the same label twice or skip a label. Gelman and Gallistel's interpretation is that children understand the principle of one-to-one correspondence, and that their errors result from mechanical difficulties in applying the principle.

A model to represent children's knowledge of counting has been developed (Note 1). One goal of this theoretical study is to clarify what it means to understand a principle. A model in the form of a computer simulation program represents knowledge in an explicit form, so it is possible to identify the knowledge structures that are responsible for the performance that is being explained. The model is designed to explain the systematic errors that children make; the model can make those same kinds of errors in a natural way. For example, an idiosyncratic list can be generated and used by the model. The stability of such a list is achieved by inclusion of a successor relation between the elements of the list. Thus, in the model, understanding of the principle of stable ordering corresponds to a process for storing lists with an explicit sequential structure. Another example involves the evidence for understanding one-to-one correspondence. Correct counting performance is achieved partly by a simple coordination of procedures; each time the list of counting tags is incremented a new object is found and counted. To explain errors that occur in partitioning, a somewhat unreliable process of perceptual grouping is used to keep track of which objects have been counted.

Semantic Knowledge

In addition to knowing how to count, children have considerable skill in comprehending quantitative information when they enter school. There have been data at least since 1930 (Buckingham & Maclatchy, 1930) showing that children can solve simple word problems before they study arithmetic, and that it is helpful to children if they have some blocks or other objects available to represent the quantities involved in problems. However, sufficient knowledge is not yet available for characterizing children's semantic knowledge about arithmetic when they enter school. Studies currently are in progress investigating the kinds of problems preschool children can solve, and how the use of objects such as blocks contributes to their success. We hope that within the next two or three years an adequate characterization of the procedures that preschool children have for representing quantitative information of the kind that is included in the solution of word problems will be formulated. The representational procedures that children have when they enter school undoubtedly develop further as they learn arithmetic, and the more fully developed representational procedures constitute important knowledge used in understanding word problems that children solve using operations and notation of formal arithmetic.

Model of Learning

While our models of children's knowledge prior to learning arithmetic and the knowledge that they acquire are incomplete in many important ways, we have some reasonable clues about the main kinds of knowledge that have to be represented in models of the initial state and the outcome of instruction in primary arithmetic. The starting state involves procedures for counting objects that include understanding of some important principles and procedures for representing the quantitative relationships involved in some simple word problems. The outcome state includes procedures for answering questions about basic facts, procedures for interpreting simple arithmetic sentences, and procedures for understanding and solving word problems using arithmetic operations.

A model of learning has the task of relating these two states and showing how the transition is made from one to the other. We are in a very early stage in the development of a learning model of arithmetic. All that can be offered at this time are some conjectures about general concepts that may be useful components of a framework for studying the process of learning. These conjectures are based on two general concepts. One is a distinction between a formal mathematical or logical language and a semantic model of the language. The other is a distinction between explicit and implicit understanding of a concept.

Formal Language and Semantic Models

The term "semantic model" in its technical mathematical sense (Tarski, 1956) refers to a set of objects and relations in which propositions of a formal language are satisfied. The distinction between a formal language and semantic interpretations of the language is useful in understanding what children must learn in their study of arithmetic. Consider the task of solving a word problem such as "Sue had three marbles; Nancy gave her five more. How many does Sue have now?" The arithmetic sentence needed is "3 + 5 = 8." We can view arithmetic as a formal language and the facts of addition and subtraction as a set of theorems that can be derived in the language. One possible semantic model of that language can be formed containing situations involving possession of different numbers of objects by persons and transfers of possession between persons. Thus a problem such as "Sue had three marbles, etc." describes a situation in the model. "3 + 5 = 8" is a sentence in the formal language. The task of solving a word problem can be seen as requiring a mapping of some semantic model of arithmetic to a sentence in the formal language of arithmetic.

An interesting possibility is that the process of counting is an essential part of children's understanding of the relation between the formal language of arithmetic and semantic models of arithmetic. To say that a set and a relation provide a semantic model for a language, we must show that sentences in the language are true in the model. For example, the sentence "3 + 5 = 8" is in the formal language, and therefore we should be able to verify that a transfer of five marbles to Sue when Sue had three to start with really does result in Sue having eight marbles. This verification requires a procedure, and the procedure is counting. Thus, counting provides an operation that maps sets of objects onto the symbols that are used in forming arithmetic sentences.

Now consider what children know when they start school, and what they must learn. The fact that preschool children can solve simple word problems (Buckingham & Maclatchy, 1930) shows that they understand some models of arithmetic. Second, children know how to count sets of objects; that is, they know the procedure for mapping entities in the model onto symbols in the formal language of arithmetic. A third component of knowledge that many children have when they enter school is the knowledge used in recognizing graphical symbols for the numbers.

What students do not know at the beginning of school instruction are the syntactic rules of arithmetic and the rules for making inferences in the formal language. For example, children are required to learn that in the sentence, "3 + 5 = 8", the major syntactic constituents are separated by the equal sign, and that in the problem "$3 + \square = 8$" the answer is found by finding the difference between the given numbers, rather than their sum. It may be quite useful to consider the process of acquiring a formal deductive language as an important part of the child's learning of arithmetic. From the fact that they can solve simple problems when they enter school, it is reasonable to assume that children have a considerable part of the semantic knowledge they need for understanding basic relationships involved in addition and subtraction. Then they are required to learn a notational system for describing these relationships in an abstract way. If one views the learning problem in this way, one can hope to make use of the considerable and growing literature on the acquisition of natural language, especially investigations of the formal properties of systems that acquire syntactic knowledge (e.g., Anderson, 1977; Hamburger & Wexler, 1975).

At the same time the first- and second-grade children are acquiring knowledge of the formal syntax of arithmetic, they also are enriching their conceptual understanding about relationships among sets of objects. Recall that this is the period in which children develop their understanding of class inclusion so that they can correctly answer questions that confused them in their preschool years (Inhelder & Piaget, 1964). They also acquire an enriched set of procedures for counting; for example they develop a flexible capability for counting on and tallying the number of counts between two numbers that they lack at the beginning of their schooling (Steffe, Hirstein, & Spikes, 1977). A desirable goal of instruction would be that children should acquire a strong set of relationships between their developing knowledge of the formal language of arithmetic syntax, their knowledge of counting procedures, and their developing knowledge about situations that constitute semantic models of arithmetic. It would be fruitful to study the relationship between formal syntactic rules, counting procedures, and semantic relationships in some detail, and to give some attention to the possibility of developing better ways of teaching the relationships between arithmetic concepts and realistic situations that are familiar to children.

In the previous discussion arithmetic has been considered as a formal language with semantic models involving sets of physical objects and events. The same general kind of theoretical analysis used here for basic arithmetic can be used to understand the relationship between arithmetic and abstract algebra. It is possible to consider arithmetic as a semantic model for more abstract deductive systems. In particular, arithmetic is a semantic model of abstract algebra. For example, "Operation *o* is commutative" is a sentence in the formal language of algebra that happens to be a true sentence about the operation of addition in arithmetic.

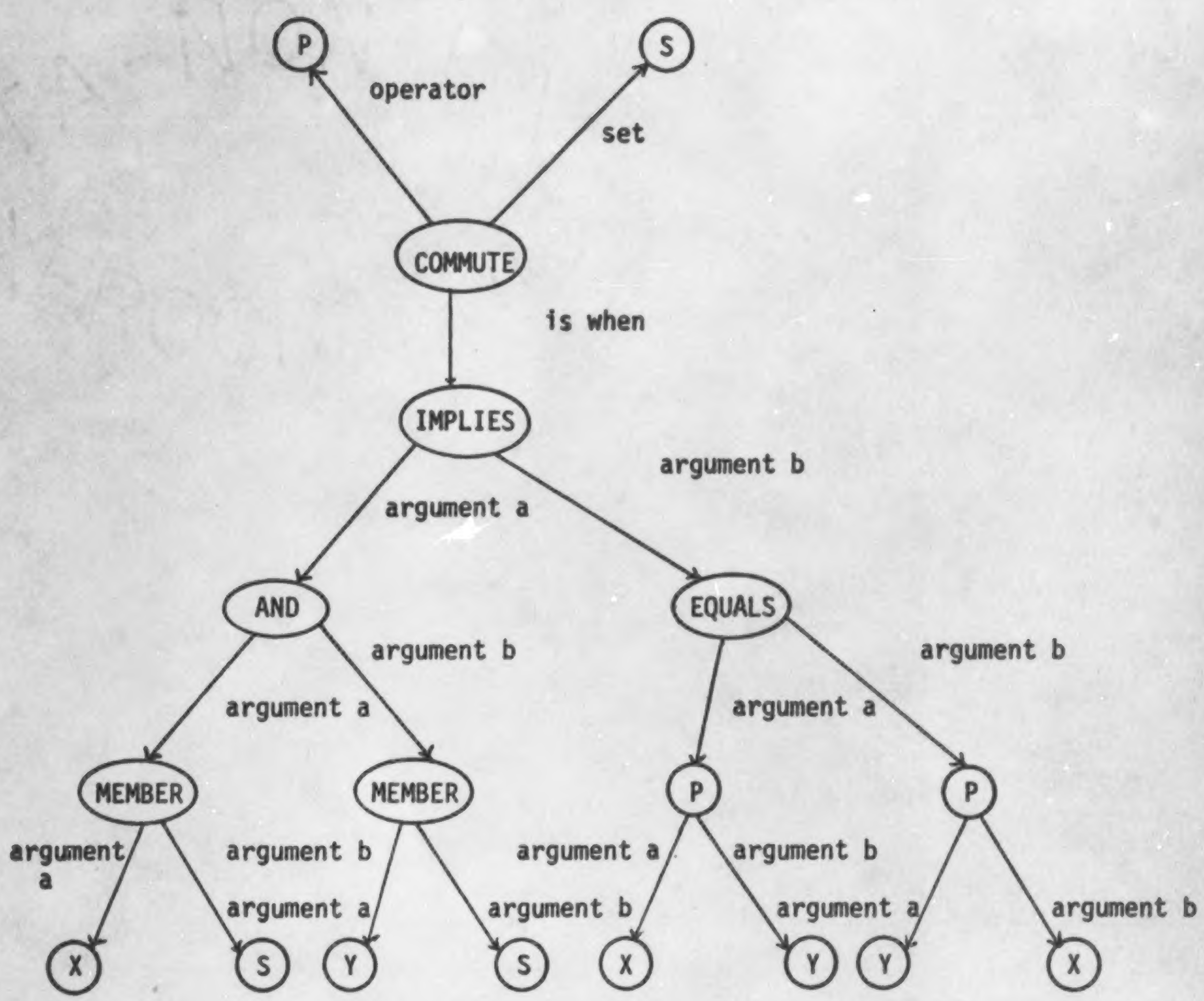
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Figure 4. Schema representing relational knowledge of the concept of commutativity.

The language of abstract algebra can be learned, just as the language of arithmetic can be learned. Figure 4 shows a semantic network representing knowledge of the meaning of commutativity. With a schema of this kind it is possible to understand a sentence such as, "Addition is commutative over the set of real numbers." In this context, properties of arithmetic can be understood using more abstract concepts of algebra, and this is analogous to the understanding of situations involving quantitative relationships using the concepts of arithmetic. Of course, it is possible to understand arithmetic without knowing algebra, in the same sense as it is possible to understand quantitative information about situations without knowing arithmetic. However, clarification of the nature of understanding in these various contexts is needed.

Explicit and Implicit Understanding

It is helpful in considering different kinds of understanding in mathematics to distinguish between explicit and implicit understanding of a concept. In order to have explicit understanding of a concept an individual must have knowledge of a formal language in which the concept occurs so that the concept can be stated and related to other concepts by stating propositions in the formal language. Implicit conceptual understanding does not require that an individual knows a formal language. Implicit knowledge of concepts does require that the procedures used for reasoning in a domain should at least have the properties that are specified by those concepts in a formal language for which the domain is a model.

It is sometimes asserted that the goals of arithmetic instruction include children's understanding of general properties of arithmetic operations, such as commutativity of addition. However, it is far from clear just what is intended by such goals. One possible meaning is that students should learn the language of abstract algebra so they can talk about arithmetic theoretically. While this might be desirable, it seems unrealistic to suppose that elementary students would acquire explicit knowledge of algebra accompanied by a reasonable degree of understanding of its relationship to arithmetic.

A second possible interpretation is that students' use of arithmetic should correctly reflect the algebraic properties of arithmetic. For example, the counting procedure shown in Figure 1 includes an operation of choosing the larger addend to start counting, and this operation would not be appropriate if addition were not commutative. While the requirement of explicit algebraic knowledge seems too strong an interpretation of the goal of general understanding, the requirement of simple conformity to algebraic properties seems too weak. Correct arithmetic procedures all conform to the algebraic properties of arithmetic, and it seems too little to ask that students merely learn the procedures correctly, if what is desired is that they understand the general properties of the operations.

Something intermediate is apparently needed, and the kind of analysis that is needed can be formulated with theoretical concepts and methods that are now available. An example is in the model of preschool children's counting that has been developed (Note 1). In that model principles of counting such as stable ordering and one-to-one correspondence are represented by general procedures that influence performance in a generative way. For example, stable ordering is represented in a procedure that generates idiosyncratic lists of counting labels. Such a procedure involves knowledge that is more general than simply knowing a list whose elements are linked together and therefore conforms to the principle of stable ordering. On the other hand, its presence in the system does not require knowledge of an abstract language in which the principle can be stated and explained. Understanding of principles in this way is implicit, since it is not supported by knowledge of a formal language; but since it can generate new instances of the concept rather than merely imitate observed instances, it includes representation of the principles in a general way, and thus goes beyond the simple requirement of conformity that could be satisfied simply by knowing how to perform correctly.

Still to be developed are models of addition and subtraction that include representations of implicit generative understanding of principles. However, this will be an important issue in further research analyzing processes involved in learning arithmetic.

Summary

Some preliminary progress toward various parts of a model of children's knowledge before they receive instruction in arithmetic and the knowledge they acquire in that instruction has been made. A model of children's counting knowledge, which they have at the beginning, has been developed, and fairly well developed models of procedural knowledge of basic addition and subtraction facts, which they have at the end have been formulated. Analyses have been made of the semantic structures involved in simple word problems that require addition and subtraction, some of which are known by preschool children and some more of which are known by preschool children and some more of which are acquired during the primary-grade years. A number of questions remain. Development of a definite characterization of the formation of semantic knowledge is needed. Also needed is an understanding of the process of comprehension of arithmetic sentences.

No results are available yet concerning the main theoretical issue of learning. On the other hand, there seem to be some potentially useful clues about the nature of that model in work that is progressing on the nature of language acquisition. In the case of mathematics learning, the language that is acquired is a formal system, and there probably are significant differences between the process of acquiring a natural language and acquiring a formal language. Indeed, one of the important questions about mathematics learning is the acquisition of understanding of relationships between the formal language of arithmetic and the structure of semantic models of the formal language. There are surely some deep questions that have only been identified and understood in a preliminary way, but current prospects seem both promising and exciting.

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INTRACONCEPT ANALYSIS OF RATIONAL NUMBERS ADDITION AND SUBTRACTION: INDIRECT VALIDATION STUDIES

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Gagné has characterized a learning hierarchy as "representing the most probable expectation of greatest positive transfer for an entire sample of learners for whom we know nothing more than what specifically relevant skills they start with" (Gagné, Note 1). Of all the learning theories proposed today, the hierarchical theory seems to be the most potentially useful for practical classroom situations. There appears to be little doubt that, in general, new skills and knowledge emerge from lower order knowledge, and that there is a significant amount of positive transfer from each successive subordinate level to the next higher level in a hierarchical ordering of such levels. From purely content considerations, students learn simple tasks first and then progress to increasingly more complex tasks. For example, mathematics content, by its very nature, has an inherent logical structure. An instructional analysis of a mathematics concept or principle would consist of a list of prerequisite mathematical learnings deemed necessary for the logical development of the concept. The analysis would yield hierarchical relationships between and among various prerequisite tasks based on the structure of the discipline. These tasks "may or may not be" ordered in such a fashion that mastery of each successively higher order skill depends in part or completely upon acquisition of the simpler prerequisite skills.

Instructional design requires decisions about structuring content and designing and ordering a set of instructional tasks. Gagné (1967) and Briggs (1968) have proposed the use of instructional sequences that require the learner to follow a specific route through a content structure. They suggest that optimal instructional sequences can be developed by sequencing materials according to learning hierarchies. Although other educators and psychologists hold this point of view (Ausubel, 1963; Bruner, 1964; Glaser, 1964; Suppes, 1966), there is little experimental evidence to support this belief.

A common strategy employed in attempting to verify the role of sequence in learning has been to generate learning hierarchies using task analysis based primarily on epistomological considerations in order to (a) examine relationships within hierarchies and/or (b) compare a logical or ordered sequence of instruction with a random or scrambled sequence. Studies examining relationships within hierarchies conducted in the sixties provide evidence to support the hierarchical structure of knowledge but produce no validated hierarchies (Gagné & Paradise, 1961; Gagné & Brown, 1961; Gagné, 1962, 1963; Gagné, Mayor, Garstens, & Paradise, 1962; Gagné & Staff, 1965). Studies conducted during this same time period in which ordered sequences of instruction were compared to scrambled sequences yielded no significant differences (Roe, Case, & Roe, 1962; Payne, Krathwhol, & Gordon, 1967; Pyatte, 1969). As Pyatte (1969) points out, a major problem with this latter type of study is determining whether the ordered sequence is in fact ordered and the random sequence random.

An examination of results from recent studies (Brown, 1970; Phillips & Kane, 1973; Callahan & Robinson, 1973) suggests that optimal learning sequences can be developed by sequencing instruction according to learning hierarchies. That is, subjects using materials sequenced according to learning hierarchies do tend to perform better with respect to time to complete the instructional program, to errors made on the programs, and to performance on a criterion test of complex problem-solving skills, than subjects using materials whose sequence was scrambled.

While the results of these latter studies are encouraging, there is some question with respect to the hierarchy validation procedures used in these studies and the studies previously cited. White (1973) claims that "the lack of a test of hierarchical dependence which takes account of errors of measurement mean that no meaningful quantitative conclusion has been reached about the validity of even one step in any hierarchy." Thus, optimal sequences of instruction based on learning hierarchies using present validation procedures are suspect.

A second, more rigorous, strategy or model for learning hierarchy research has been discussed by Uprichard (1970) and White (1973). The effect of sequence on learning is studied by focusing on two or three concepts or skills in a given hypothesized hierarchy rather than the entire hierarchy. This technique yields a fixed-effects design for the independent variable-instructional sequence. White (1973) describes this process more specifically as follows:

> Let the Roman numerals I, II, III,....stand for elements which compose a postulated learning hierarchy. Suppose that it has been hypothesized that possession of II is essential for the learning of I. Then the rigorously logical method testing the truth of the hypothesis is:
>
> 1. Secure a sample of Ss.
> 2. Test their possession of element I..
> 3. Discard those who have II, and divide the remainder randomly into two groups.
> 4. Teach one group element II, then I; teach the other group element I only.
> 5. Test both groups for possession of I and II. If no incidental learning has taken place, none of the second group will possess II, and if the hypothesis is correct only the first group will contain members who possess I. Some members of the second group may acquire II through being exposed to the teaching of I, and some of them may have acquired I also. But if the hypothesis is correct,

> none of the second group will possess I without II (pp. 317-72).

This strategy has been used successfully by researchers to establish hierarchical relationships between two or three elements (Scandura, 1966; Uprichard, 1970; Resnick, Siegel, & Kresh, Note 3), but becomes impractical when studying hierarchies containing four or more levels. Hence, its application to acquisition of school related knowledge (mathematics and reading) is severely limited.

The two strategies reviewed to this point directly involve both the testing and instruction of subjects to determine an optimal sequence of instruction based on a learning hierarchy. A third strategy that is emerging in the literature is an indirect hierarchy validation procedure where subjects are tested but not instructed (Miller, 1969; Uprichard & Phillips, 1975, 1976; Causey & Underhill, 1976; Hill & Underhill, Note 2). The steps followed in this strategy are:

1. A conceptual schema or model is used to hypothesize a learning hierarchy.
2. For each hypothesized level of the hierarchy a test item or task is constructed.
3. The items or tasks are administered to a large sample of subjects (N > 100.).
4. Data is collected and subjected to hierarchy validation procedures such as the Walbesser Method (Walbesser, 1968) and/or Pattern Analysis (Remoldi & Grib, 1960).
5. The original hypothesized hierarchy is examined in terms of the data collected and inferences made about the conceptual schema used to generate the hierarchy.

The purpose of the indirect validation strategy is not to establish the optimal sequence of instruction based on a given validated learning hierarchy but rather to "refine" or "fine tune" the conceptual schema used to generate the hierarchy and gain some hunches about an optimal or more efficient sequence for teaching the content studied. These educated hunches can facilitate the development of diagnostic and perscriptive instructional programs. For the most part researchers using this strategy do not believe there is one optimal instructional sequence for acquiring specific content for all learners but many such sequences can be derived. The application of results from this type of research to the classroom is both to provide the teacher with a conceptual schema/model with which to organize one's way of thinking about teaching content and to give some hunches about where to begin in the system and how to move through it. The major contribution of this type of research is the "refined" conceptual schema. Without such a detailed schema, diagnostic and prescriptive instruction is almost impossible to implement. However, in interpreting results from research using the indirect validation strategy, one must be aware of limitations due to possible confounding variables such as prior educational experience of subjects and errors of measurement.

Of the three strategies presented above, which one ought researchers interested in instructional sequences be spending time on? The answer to this question depends on the researchers' goal. If one believes that there is one optimal instructional sequence for acquiring specific content and that the sequence is based on a validated learning hierarchy containing many levels, then the first strategy discussed, used by Gagné and others, would be appropriate. Although this line of research has made a contribution to the field of education from a conceptual standpoint, there is little experimental evidence to support any validated learning hierarchy. If one is interested in studying hierarchical relationships from a more theoretical rather than practical framework, then the second strategy where two or three concepts or skills in a hierarchy are examined rather than the entire hierarchy is useful. Finally, if one believes that there is more than one optimal sequence for acquiring specific content, then the indirect validation strategy is most appropriate. Here, the emphasis is placed on designing and refining a conceptual schema for generating optimal sequences and providing hunches about how to use it in designing an instructional sequence. This third strategy might be the most efficient in terms of time, effort, and payoff for classroom teachers.

The two studies briefly summarized below involve an indirect validation approach. Each summary will include a description of the conceptual schema used to generate optimal sequences and some hunches to consider in using it.

Intraconcept Analysis Of Rational Number Addition

The investigators in this study were concerned with the generation and validation of an hypothesized learning hierarchy. More specifically, the purposes of the study were (1) to develop a learning hierarchy for rational number addition using the intraconcept analysis technique, and (2) to test the validity of the hypothesized hierarchy using the Walbesser Technique and Pattern Analysis. The objective in doing an intraconcept analysis is to generate a series of tasks each of which represent an operationalized level of a given concept or principle. In this type of analysis, the first step is to identify a set of dimensions that can be used to define the concept or principle selected. The dimensions associated with particular content should reflect related but non-overlapping aspects of the content. For example, in the intraconcept analysis on addition of rationals in fraction form, the two dimensions identified were number theory and renaming. Number theory (primes and composites) was used to operationally define denominator classes. Categories within a denominator class were defined by the renamings required in the sum (answer). The end result of an intraconcept analysis is a conceptual schema that defines levels of the content to be taught. Although the levels are somewhat structured they may or may not be hierarchical in nature.

The intraconcept analysis technique differs from the more widely used task analysis model in that consideration is given to both psychological and content (discipline) factors in developing and hierarchically ordering tasks. Because of this, the authenticity of many tasks generated using this analysis may be questioned by content area specialists. However, many educators (including the authors), who have worked in clinical situations with young children experiencing difficulty in learning mathematics, question whether one can simply impose mathematical or adult logic on children. This study was conducted in hopes of shedding some light on the epistemological-psychological balance needed in developing efficient or optimal instructional sequences in at least one area of mathematics.

Procedure

Development of hierarchy. The specific strategy used in the intraconcept analysis of addition of rationals in fraction form was as follows:

a) Problems were divided into two levels: Like Denominators and Unlike Denominators. Classes within each level were identified by the nature of (prime and composite) and relationship between (multiple or not) denominators of the two rationals added. The classes, in their hypothesized order of difficulty, were:

Like Denominators

I. Composites. (1/4 + 2/4 = ___)
II. Primes. (1/5 + 1/5 = ___)

Unlike Denominators

III. Composites- one composite multiple of other. (1/4 + 1/8 = ___)
IV. Prime, composite- composite a multiple of prime. (1/5 + 1/10 = ___)
V. Primes. (1/5 + 1/7 = ___)
VI. Prime, composite- composite not multiple of prime. (1/5 + 1/9 = ___)
VII. Composites- not multiples. (1/6 + 1/8 = ___)

b) Within each denominator class an attempt was made to generate five renaming categories.[1] The renaming categories were operationally defined in terms of renaming required in the sum. The five categories in their hypothesized order of difficulty were:

I. Sum- proper fraction in simplest form without renaming. (1/4 + 2/4 = 3/4)
II. Sum- improper fraction renamed to one (1). (1/4 + 3/4 = 4/4 = 1)

---

[1] Five renaming categories do not exist in some denominator classes.

III. Sum- proper fraction renamed to simplest form. (1/4 + 1/4 = 2/4 = 1/2)
IV. Sum- improper fraction renamed to mixed numeral in simplest form, one renaming. (3/4 + 2/4 = 5/4 = 1 1/4)
V. Sum- improper fraction renamed to mixed numeral in simplest form, two renamings. (3/4 + 3/4 = 6/4 = 1 2/4 = 1 1/2)

c) Two tasks involving mixed numerals (whole number sums less than or equal to nine) were associated with each renaming category within a denominator class. A renaming category and its two related tasks are presented below.

Renaming Category I	Task #1	Task #2
1/4 + 2/4 = ___	1 1/4 + 2/4 = ___	1 1/4 + 1 2/4 = ___

Content area specialists might question the idea of differentiating between primes and composites in generating the six denominator classes in this analysis. Indeed they could logically argue that there should be but four classes: (I and II), (III and IV), (V and VI), and VII. Questions could also be raised in regard to the differentiations made among renaming categories.

An intraconcept analysis of rational number addition for denominator class I - Like Composite is presented in Figure 1.

Analysis of hierarchy. The procedure as described yielded eighty-nine specific tasks. In order to hold testing to a manageable size, some tasks within classes were collapsed. The final hierarchy consisted of 45 tasks including (a) all existing sum categories within identified classes and (b) a sampling of tasks involving mixed numerals within classes. Based upon the hypothesized ordering of the subordinate tasks, a test was constructed to assess mastery at each level in the hierarchy. "Pass" was defined as correct sum expressed in simplest form. For ease in computation the tasks appeared in vertical form.

The test was administered to 251 students in grades 4 through 8 in order to obtain a wide range of ability levels. The majority of the subjects were in grades 5 and 6. Subjects who passed or failed all 45 items were excluded from the study since inferences about the order of items cannot be based on responses from these students (Phillips & Kane, 1975). The resulting sample contained 200 subjects.

The patterns of responses for each transfer in the hierarchy were analyzed using both the Walbesser Technique (Walbesser, 1968) and Pattern Analysis (Rimoldi & Grib, 1960). The Walbesser Technique is based on the 2 x 2 contingency table of pass-fail responses (Figure 2).

I	II	III	IV	V
$\begin{array}{r} \frac{2}{6} \\ +\ \frac{3}{6} \\ \hline \frac{5}{6} \end{array}$	$\begin{array}{r} \frac{4}{6} \\ +\ \frac{2}{6} \\ \hline \frac{6}{6} = 1 \end{array}$	$\begin{array}{r} \frac{1}{6} \\ +\ \frac{1}{6} \\ \hline \frac{2}{6} = \frac{1}{3} \end{array}$	$\begin{array}{r} \frac{5}{6} \\ +\ \frac{2}{6} \\ \hline \frac{7}{6} = 1\ \frac{1}{6} \end{array}$	$\begin{array}{r} \frac{5}{6} \\ +\ \frac{5}{6} \\ \hline \frac{10}{6} = 1\ \frac{4}{6} = 1\ \frac{2}{3} \end{array}$
$\begin{array}{r} 3\ \frac{2}{6} \\ +\ \frac{3}{6} \\ \hline 3\ \frac{5}{6} \end{array}$	$\begin{array}{r} 3\ \frac{4}{6} \\ +\ \frac{2}{6} \\ \hline 3\ \frac{6}{6} = 4 \end{array}$	$\begin{array}{r} 3\ \frac{1}{6} \\ +\ \frac{1}{6} \\ \hline 3\ \frac{2}{6} = 3\ \frac{1}{3} \end{array}$	$\begin{array}{r} 5\ \frac{5}{6} \\ +\ \frac{2}{6} \\ \hline 2\ \frac{7}{6} = 3\ \frac{1}{6} \end{array}$	$\begin{array}{r} 3\ \frac{5}{6} \\ +\ \frac{5}{6} \\ \hline 3\ \frac{10}{6} = 4\ \frac{4}{6} = 4\ \frac{2}{3} \end{array}$
$\begin{array}{r} 3\ \frac{2}{6} \\ +4\ \frac{3}{6} \\ \hline 7\ \frac{5}{6} \end{array}$	$\begin{array}{r} 3\ \frac{4}{6} \\ +5\ \frac{2}{6} \\ \hline 8\ \frac{6}{6} = 9 \end{array}$	$\begin{array}{r} 3\ \frac{1}{6} \\ +4\ \frac{1}{6} \\ \hline 7\ \frac{2}{6} = 7\ \frac{1}{3} \end{array}$	$\begin{array}{r} 2\ \frac{5}{6} \\ +3\ \frac{2}{6} \\ \hline 5\ \frac{7}{6} = 6\ \frac{1}{6} \end{array}$	$\begin{array}{r} 3\ \frac{5}{6} \\ +2\ \frac{5}{6} \\ \hline 5\ \frac{10}{6} = 6\ \frac{4}{6} = 6\ \frac{2}{3} \end{array}$

Figure 1. INTRACONCEPT ANALYSIS - ADDITION OF RATIONALS IN FRACTION FORM (Like Composite Denominator)

		Item 1	
		fail (-)	pass (+)
Item 2	pass (+)	-+	++
	fail (-)	--	+-

Figure 2. Contingency Table

Using the frequency of students falling in each cell shown in Figure 2, the following three ratios were computed for every possible relationship in the hierarchy (i.e., level 1 with 2, 1 with 3, ... 1 with 45; 2 with 3, 2 with 4, ... 2 with 45, etc.).

(1) Consistency Ratio $= \frac{(++)}{(++) + (+-)}$

The value of this ratio is a measure of how consistent the data are with the hypothesized dependency.

(2) Adequacy Ratio $= \frac{(++)}{(++) + (-+)}$

The value of this ratio is a measure of the adequacy of the identified levels.

(3) Completeness Ratio $= \frac{(++)}{(++) + (--)}$

The value of this ratio is a measure of the effectiveness of instruction.

The level of acceptability used for each of these ratios was that determined by Phillips (1971) instead of those proposed by Walbesser since no instructional sequences were involved. These levels were: (1) Consistency Ratio .85, (2) Adequacy Ratio .70, (3) Completeness Ratio .50.

The pattern analysis technique was used to analyze the responses for the complete hierarchy on a subject by subject basis. The index of agreement given by the pattern analysis indicates the amount of agreement or correlation between two patterns. In this case, the index of agreement indicates the agreement between the observed and expected patterns. If the tasks were truly hierarchical, where each subtask was a necessary prerequisite to the next, once a learner failed a given level he would be expected to fail all subsequent levels. Thus, the expected pattern was defined as one where no correct responses followed an incorrect response.

Results. The initial hypothesized hierarchy developed using an intraconcept analysis is given in Table 1. A computer program based on the Walbesser Technique was used to give the pass-fail response patterns between all relationships. That is, item 1 was paired with all 45 items; item 2 with all items; etc.; until all possible pairs of items were considered.

In order to analyze the hypothesized hierarchical sequence, the consistency, adequacy, and completeness ratios for each relationship within the hierarchy were examined. No ordering of the 45 tasks yielded acceptable levels in all three ratios. The ordering which yielded the best fit to the data is shown in Table 2.

The pattern of responses obtained from the hypothesized ordering of subordinate tasks yielded an index of agreement of .68. After final revision, the empirical sequence yielded an index of agreement of .70 which indicates a slightly higher agreement between the expected and observed response patterns. No statistical test of significance for the index of agreement has been developed.

The internal consistency of the test based on the initial hierarchy was determined using the Kuder-Richardson Formula 20 (Nunnally, 1967). The value of this coefficient was .98. The pattern of responses for the empirical sequence was analyzed to determine if the ordering exhibited a hierarchical structure based on item difficulty. Item difficulty of each task in the ordered sequence is given in Table 3. Although there would be some discrepancies between a hierarchical sequence based on item difficulty and the ordered sequence determined, the general pattern of item difficulty of the latter is acceptable. Further, item difficulty alone is not considered an adequate technique for determining hierarchical relationships (Phillips, 1971).

Since many task comparisons involving mixed numerals resulted in low Walbesser ratios (see Table 2), an empirical sequence without mixed numerals was determined. The index of agreement for this sequence which is presented in Table 4 was .75.

Discussion and Implications

An examination of data in Table 2 (Walbesser ratios) indicates that with few exceptions task comparisons meet specified criteria on two of the three ratios. The consistency ratio was met in 35 of 44 comparisons, the adequacy ratio in 43 of 44, and the completeness ratio in only 27 of 44. It should be noted, however, that the first two ratios reported are more critical when determining a hierarchy from test data. A considerable improvement in the number of comparisons meeting criteria on ratios was observed for the hierarchical sequence determined when tasks involving mixed numerals were omitted (Consistency, 26/28; Adequacy, 28/28; and Completeness, 20/28; see Table 4). The index of agreement for the twenty-nine task hierarchy (no mixed numerals) was greater than that of the forty-five task hierarchy .75 vs .70. This difference may be directly related to the number of tasks in each hierarchy.

TABLE 1

Initial Hypothesized Hierarchy

1. 1/4 + 2/4 = ___
2. 1/5 + 3/5 = ___
3. 2 1/6 + 4/6 = ___
4. 3 1/8 + 4 2/8 = ___
5. 1/4 + 3/4 = ___
6. 3/7 + 4/7 = ___
7. 1/6 + 1/6 = ___
8. 5/8 + 4/8 = ___
9. 2/3 + 2/3 = ___
10. 4 3/5 + 4/5 = ___
11. 5 3/7 + 1 5/7 = ___
12. 3/4 + 3/4 = ___
13. 2 3/6 + 5/6 = ___
14. 6 5/8 + 2 5/8 = ___
15. 1/3 + 1/9 = ___
16. 1/4 + 1/8 = ___
17. 2 1/12 + 2/4 = ___
18. 3 1/8 + 5 1/16 = ___
19. 1/3 + 1/5 = ___
20. 1/5 + 8/10 = ___
21. 3/4 + 3/12 = ___
22. 1/3 + 2/6 = ___
23. 1/4 + 2/3 = ___
24. 3 2/5 + 2/10 = ___
25. 4 1/3 + 3 3/9 = ___
26. 2/3 + 3/6 = ___
27. 5/8 + 7/10 = ___
28. 2/3 + 3/5 = ___
29. 5 3/5 + 3/7 = ___
30. 3 1/2 + 4 2/3 = ___
31. 2/3 + 6.9 = ___
32. 3/4 + 4/8 = ___
33. 1/3 + 1/10 = ___
34. 2/6 + 3/7 = ___
35. 1/4 + 1/6 = ___
36. 3/6 + 4/3 = ___
37. 6 4/8 + 5/10 = ___
38. 5 2/4 + 2 3/6 = ___
39. 1/6 + 3/10 = ___
40. 1/3 + 7/10 = ___
41. 3/8 + 11/12 = ___
42. 4/6 + 4/7 = ___
43. 3 2/3 + 5/10 = ___
44. 4 3/5 + 3 4/8 = ___
45. 3/4 + 3/6 = ___

TABLE 2

Walbesser's Ratios for the Empirical Ordering (N=200)

Level	Consistency	Adequacy	Completeness
1-2	.98	.96	.97
*2-4	.95	.76	.97
*4-5	.85	.84	.83
5-6	.99	.99	.76
6-8	.98	.84	.73
8-9	.97	.98	.65
9-7	.86	.85	.69
*7-11	.87	.79	.66
*11-3	.70	.70	.67
*3-10	.72	.60	.60
*10-16	.78	.80	.56
16-15	.95	.93	.59
15-33	.94	.88	.58
33-19	.93	.94	.55
19-35	.93	.91	.54
35-20	.94	.88	.55
20-21	.96	.94	.56
21-36	.97	.90	.53
36-12	.83	.83	.51
*12-14	.90	.90	.51
*14-13	.94	.96	.51
*13-23	.81	.88	.52
23.22	.89	.92	.53
22-39	.94	.80	.48
39-34	.81	.88	.41
34.27	.98	.72	.47
27-26	.95	.97	.54
26-40	.95	.90	.52
40-28	.85	.89	.49
28-41	.89	.80	.46
41-31	.92	.82	.46
31-32	.93	.95	.50
32-42	.93	.79	.46
*42-18	.90	.73	.48
*18-25	.93	.76	.48
*25-17	.60	.78	.35
*17-24	.61	.73	.32
*24-30	.77	.94	.44
*30-38	.94	.92	.48
*38-44	.88	.81	.41
*44-29	.88	.79	.42
*29-37	.87	.81	.42
*43-45	.74	.85	.36

*indicates comparison involving mixed numerals

TABLE 3

Item Difficulty-Empirical Hierarchy (45 Tasks)

Task	Difficulty	Task	Difficulty
1	.94	31	.54
2	.91	32	.53
*4	.74	42	.43
5	.76	* 18	.55
6	.73	* 25	.45
8	.64	* 17	.35
9	.66	* 24	.42
7	.64	* 30	.51
*11	.59	* 38	.50
*3	.61	* 44	.42
*10	.48	* 29	.47
16	.59	* 37	.46
15	.59	* 43	.38
33	.57	45	.41
19	.57		
35	.54		
20	.58		
21	.55		
36	.53		
12	.51		
*14	.51		
*13	.50		
23	.55		
22	.53		
39	.46		
34	.42		
27	.56		
26	.53		
40	.51		
28	.50		
41	.47		

*indicates problems involving mixed numerals

TABLE 4

Walbesser's Ratios for the Empirical Ordering- No Mixed Numerals (N=200)

Level	Consistency	Adequacy	Completeness
1-2	.98	.96	.97
2-5	.95	.76	.96
5-6	.99	.99	.76
6-8	.98	.84	.73
8-9	.97	.98	.65
9-7	.86	.85	.69
7-12	.95	.74	.60
12-16	.76	.86	.56
16-15	.95	.93	.59
15-33	.94	.88	.58
33-19	.93	.94	.55
19-35	.93	.91	.54
35-20	.94	.88	.55
20-21	.96	.94	.56
21-36	.97	.90	.53
36-23	.92	.87	.53
23-22	.89	.92	.53
22-39	.94	.80	.48
39-34	.81	.88	.41
34-27	.98	.72	.47
27-26	.95	.97	.54
26-40	.95	.90	.52
40-28	.85	.89	.49
28-41	.89	.80	.46
41-31	.92	.82	.46
31.32	.93	.95	.50
32-45	.93	.79	.46
42-45	.86	.81	.40

The empirically determined sequence is presented in Table 5 in terms of the defining characteristics of each task. An examination of the sequence tends to support the following hunches for teaching youngsters the addition of rationals in fraction form.

1. Tasks involving like denominators should be taught before those involving unlike denominators, with one exception, namely, sum category V (see Table 5).

2. In working with like denominators, one should teach tasks within a given sum category before proceeding to the next one. The sum categories in hierarchical order are I, II, IV, III, V.

3. Within a sum category, tasks involving like-composite denominators should be taught before those with like-prime denominators.

4. In working with unlike denominators, one should teach tasks within a given sum category before proceding to the next one. The sum categories in hierarchical order are I, II, III, IV, V.

5. Tasks involving adding with mixed numerals should not be introduced early in the development of rational number addition skills.

6. Tasks involving adding with two mixed numerals within a denominator class and sum category should be taught before those tasks involving one mixed numeral.

7. Sequencing of unlike denominator classes is difficult and not completely determined by this study. However, a possible teaching sequence might be III, IV, VI, V, VII.

Intraconcept Analysis of Rational Number Subtraction

This study was a follow-up to the Uprichard and Phillips (1975) work described above in that its purpose was (1) to develop a learning hierarchy for rational number subtraction using intraconcept analysis, and (2) to validate that hierarchy. Again, the major levels of the hierarchy were identified according to structural relationships between denominators. Within levels, sublevels were identified according to the number skills required to complete the tasks. Skills required to complete a particular task were operationally defined from both a mathematical and psychological point of view.

Procedure

Development of hierarchy. The intraconcept analysis strategy used to analyze the subtraction of rationals in fraction form was as follows:

TABLE 5

Empirical Order of Tasks by Sum & Denominator's

DENOMINATOR CLASSES		Sum Categories				
		Proper fraction simplest form, without measuring	Improper fraction, renamed to one	Proper fraction, renamed to simplest form	Improper fraction, renamed to mixed numeral simplest form, one renaming	Improper fraction, renamed to mixed numeral simplest form, two renaming
		I	II	III	IV	V
LIKE	I. Composites	1	4	8	6	20
	a. One mixed numeral	10				22
	b. Two mixed numerals	3				21
	II. Primes	2	5		7	
	a. One mixed numeral				11	
	b. Two mixed numerals				9	
UNLIKE	III. Composites - one composite multiple of other.	12	18	23	27	33
	a. One mixed numeral	37				
	b. Two mixed numerals	35				
	IV. Prime, Composite - composite multiple of prime.	13	17	24	28	32
	a. One mixed numeral			38		
	b. Two mixed numerals			36		
	V. Primes	15			30	
	a. One mixed numeral				42	
	b. Two mixed numerals				39	
	VI. Prime, Composite - composite not multiple of prime.	14		26	29	34
	a. One mixed numeral					43
	b. Two mixed numerals					41
	VII. Composites - not multiples	16	19	25	31	45
	a. One mixed numeral		37			
	b. Two mixed numerals		40			

*Numerals in table do not correspond to test items.

a) Problems were divided into two levels - like denominators and unlike denominators. Classes within each level were identified by the nature of (prime and composite) and relationship between (multiple or not) denominators of the two rationals subtracted. The classes, in their hypothesized order of difficulty, were:

Like Denominators
A. Composites. (2/4 - 1/4 = ___)
B. Primes. (2/5 - 1/5 = ___)

Unlike Denominators
C. Composites - one composite multiple of other. (1/4 - 1/8 = ___)
D. Prime, composite - composite a multiple of prime. (1/5 - 1/10 = ___)
E. Primes. (1/5 - 1/7 = ___)
F. Prime, composite - composite not multiple of prime. (1/5 - 1/9 = ___)
G. Composites - not multiples, common factor. (1/6 - 1/8 = ___)

b) Within each denominator class an attempt was made to generate five renaming categories. The renaming categories were operationally defined in terms of renamings required in the difference and/or sum (minuend).[2]

I. Difference - proper fraction in simplest form without renaming. (2/4 - 1/4 = 1/4)
II. Difference - proper fraction renamed to zero. (1/4 - 1/4 = 0/4 = 0)
III. Difference - proper fraction renamed to simplest form. (3/4 - 1/4 = 2/4 = 1/2)
IV. Renaming in sum, difference - proper fraction in simplest form without renaming. (1 1/4 - 2/4 = 5/4 - 2/4 = 3/4)
V. Renaming in sum, difference - proper fraction renamed to simplest form. (1 1/4 - 3/4 = 5/4 - 3/4 = 2/4 = 1/2)

c) Two tasks involving mixed numerals (whole numbers greater than one and less than ten) were associated with each renaming category within a denominator class. A renaming category and its two related tasks are presented below.

Renaming Category I	Task #1 Difference of two mixed numbers	Task #2 Difference of mixed number and proper fraction
2/4 - 1/4 = ___	3 2/3 - 2 1/4 = ___	3 2/4 - 1/4 = ___

---

[2] Five remaining categories do not exist in some denominator classes.

d) Nine additional subtraction problems with whole number sums were included in the hierarchy. These problems are:

Composite Denominator, Renaming Category I	Related Tasks
	3 - 1 1/4 = 2 4/4 - 1 1/4 = 1 3/4
1 - 1/4 = 3/4	3 - 1/4 = 2 4/4 - 1/4 = 2 3/4

Composite Denominator, Renaming Category II	Related Tasks
	3 - 1 2/4 = 2 4/4 - 1 2/4 = 1 2/4 = 1 1/4
1 - 2/4 = 2/4 = 1/2	3 - 2/4 = 2 4/4 - 2/4 = 2 2/4 = 2 1/2

Prime Denominator, Renaming Category I	Related Tasks
	3 - 1 2/3 = 2 3/3 - 1 2/3 = 1 1/3
1 - 2/3 = 1/3	3 - 2/3 = 2 3/3 - 2/3 = 2 1/3

An intraconcept analysis of rational number subtraction for denominator class I - Like Composite and Related Whole Number Sum Problems is presented in Figure 3.

The above procedures yielded ninety-six specific tasks. In order to hold testing to a manageable size, some tasks within classes were collapsed. The final hierarchy consisted of 47 tasks. Based upon the hypothesized ordering of the subordinate tasks, a test was constructed to assess mastery at each level in the hierarchy. "Pass" was defined as correct difference expressed in simplest form. For ease in computation the tasks appeared in vertical form. The test was administered to 200 students in grades 4 through 8 in order to obtain a wide range of ability levels. The majority of the subjects were in grades 5 and 6. As with the addition study, subjects who passed or failed all 47 items were dropped. The resulting sample contained 174 subjects. The patterns of responses for each transfer in the hierarchy were analyzed using both the Walbesser Technique (Walbesser, 1968) and Pattern Analysis (Rimoldi & Grib, 1960).

Results

The initial hypothesized hierarchy developed using an intraconcept analysis is given in Table 6. The data analysis procedures that were used in the addition study were applied to the subtraction data. The ordering which yielded the best fit to the data is shown in Table 7.

The pattern of responses obtained from the hypothesized ordering of subordinate tasks yielded an index of agreement of .63. After final revision, the empirical sequence yielded an index of agreement of .75 which indicates a higher agreeement between the expected and observed response patterns. The internal consistency of the test based on the initial hierarchy was determined using the Kuder-Richardson Formula 20 (Nunnally, 1967). The value of this coefficient was .96. Item difficulty of each task in the ordered sequence is given in Table 8.

I	II	III	IV	V		
$\begin{array}{r}\frac{3}{4}\\ -\frac{2}{4}\\ \hline \frac{1}{4}\end{array}$	$\begin{array}{r}\frac{3}{4}\\ -\frac{3}{4}\\ \hline \frac{0}{4}=0\end{array}$	$\begin{array}{r}\frac{3}{4}\\ -\frac{1}{4}\\ \hline \frac{2}{4}=\frac{1}{2}\end{array}$	$\begin{array}{rcr}1\frac{1}{4} & = & \frac{5}{4}\\ -\frac{2}{4} & = & -\frac{2}{4}\\ \hline & & \frac{3}{4}\end{array}$	$\begin{array}{rcr}1\frac{1}{4} & = & \frac{5}{4}\\ -\frac{3}{4} & = & -\frac{3}{4}\\ \hline & & \frac{2}{4}=\frac{1}{2}\end{array}$	$\begin{array}{rcr}1 & = & \frac{4}{4}\\ -\frac{1}{4} & = & -\frac{1}{4}\\ \hline & & \frac{3}{4}\end{array}$	$\begin{array}{rcr}1 & = & \frac{4}{4}\\ -\frac{2}{4} & = & -\frac{2}{4}\\ \hline & & \frac{2}{4}=\frac{1}{2}\end{array}$
$\begin{array}{r}3\frac{3}{3}\\ -\frac{2}{4}\\ \hline 3\frac{1}{4}\end{array}$	$\begin{array}{r}3\frac{3}{4}\\ -\frac{3}{4}\\ \hline 3\frac{0}{4}=3\end{array}$	$\begin{array}{r}3\frac{3}{4}\\ -\frac{1}{4}\\ \hline 3\frac{2}{4}=3\frac{1}{2}\end{array}$	$\begin{array}{rcr}3\frac{1}{4} & = & 2\frac{5}{4}\\ -\frac{2}{4} & = & -\frac{2}{4}\\ \hline & & 2\frac{3}{4}\end{array}$	$\begin{array}{rcr}3\frac{1}{4} & = & 2\frac{5}{4}\\ -\frac{3}{4} & = & -\frac{3}{4}\\ \hline & & 2\frac{2}{4}=2\frac{1}{2}\end{array}$	$\begin{array}{rcr}3 & = & 2\frac{4}{4}\\ -\frac{1}{4} & = & -\frac{1}{4}\\ \hline & & 2\frac{3}{4}\end{array}$	$\begin{array}{rcr}3 & = & 2\frac{4}{4}\\ -\frac{2}{4} & = & -\frac{2}{4}\\ \hline & & 2\frac{2}{4}=2\frac{1}{2}\end{array}$
$\begin{array}{r}3\frac{3}{4}\\ -1\frac{2}{4}\\ \hline 2\frac{1}{4}\end{array}$	$\begin{array}{r}3\frac{3}{4}\\ -1\frac{3}{4}\\ \hline 2\frac{0}{4}=2\end{array}$	$\begin{array}{r}3\frac{3}{4}\\ -1\frac{1}{4}\\ \hline 2\frac{2}{4}=2\ _{2}\end{array}$	$\begin{array}{rcr}3\frac{1}{4} & = & 2\frac{5}{4}\\ -1\frac{2}{4} & = & -1\frac{2}{4}\\ \hline & & 1\frac{3}{4}\end{array}$	$\begin{array}{rcr}3\frac{1}{4} & = & 2\frac{5}{4}\\ -1\frac{3}{4} & = & -1\frac{3}{4}\\ \hline & & 1\frac{2}{4}=1\frac{1}{2}\end{array}$	$\begin{array}{rcr}3 & = & 2\frac{4}{4}\\ -1\frac{1}{4} & = & -1\frac{1}{4}\\ \hline & & 1\frac{3}{4}\end{array}$	$\begin{array}{rcr}3 & = & 2\frac{4}{4}\\ -1\frac{2}{4} & = & -1\frac{2}{4}\\ \hline & & 1\frac{2}{4}=1\frac{1}{2}\end{array}$

Figure 3. INTRACONCEPT ANALYSIS - SUBTRACTION OF RATIONALS IN FRACTION FORM (Like Composite Denominator) AND RELATED WHOLE NUMBER SUM PROBLEMS

TABLE 6

Initial Hypothesized Hierarchy

---

1. 3/4 - 2/4 = ___
2. 2/3 - 1/3 = ___
3. 5/8 - 5/8 = ___
4. 2/5 - 2/5 = ___
5. 3/6 - 1/6 = ___
6. 3 2/6 - 2 1/6 = ___
7. 3 4/7 - 2 3/7 = ___
8. 1 - 1/8 = ___
9. 1 - 2/6 = ___
10. 4 - 1/6 = ___
11. 5 - 2/4 = ___
12. 3 - 1 1/4 = ___
13. 6 - 1 2/10 = ___
14. 1 1/4 - 2/4 = ___
15. 1 1/7 - 2/7 = ___
16. 1 1/6 - 3/6 = ___
17. 1/4 - 1/8 = ___
18. 1/3 - 1/9 = ___
19. 1/2 - 1/3 = ___
20. 1/3 - 1/8 = ___
21. 1/6 - 1/8 = ___
22. 1/8 - 2/16 = ___
23. 4/14 - 2/7 = ___
24. 5/10 - 4/8 = ___
25. 5/12 - 1/6 = ___
26. 2/5 - 2/10 = ___
27. 6/9 - 1/2 = ___
28. 3/4 - 3/6 = ___
29. 3 1/8 - 2 1/16 = ___
30. 6 3/9 - 3 1/3 = ___
31. 8 4/8 - 2 1/3 = ___
32. 4 1/6 - 1/12 = ___
33. 3 1/5 - 2/10 = ___
34. 5 3/5 - 3/9 = ___
35. 1 1/8 - 1/4 = ___
36. 1 1/9 - 1/3 = ___
37. 1 1/5 - 2/3 = ___
38. 1 2/5 - 4/9 = ___
39. 1 1/8 - 1/6 = ___
40. 1 3/18 - 3/9 = ___
41. 1 1/5 - 4/10 = ___
42. 1 6/9 - 4/5 = ___
43. 1 2/6 - 6/9 = ___
44. 4 1/5 - 2 2/7 = ___
45. 6 1/6 - 3 2/4 = ___
46. 2 1/3 - 2/5 = ___
47. 2 1/3 - 2/5 = ___

TABLE 7

Walbesser's Ratios For The Empirical Ordering (N = 174)

Level	Consistency	Adequacy	Completeness
1-2	.99	.99	1.00
2-3	.99	.97	1.00
3-4	1.00	1.00	.97
4-6	.97	.91	.99
6-7	.94	.99	.96
7-8	.98	.87	.97
8-17	.90	.97	.95
17-18	.97	.95	.95
18-19	.95	1.00	.95
19-20	1.00	.96	.95
20-21	.97	.94	.94
21-22	.94	.92	.93
22-23	.99	.97	.88
23-24	.97	.94	.87
24-29	.90	.90	.90
29-30	.94	.93	.86
30-32	.94	.94	.86
32-33	.96	.92	.83
33-5	.83	.85	.86
5-25	.89	.85	.81
25-26	.84	.87	.81
26-27	.89	.83	.79
27-28	.88	.92	.76
28-14	.84	.84	.81
14-15	.92	.94	.78
15-16	.96	.89	.76
16-35	.89	.90	.75
35-36	.93	.92	.74
36-37	.89	.87	.74
37-9	.83	.84	.74
9-12	.88	.71	.66
12-10	.95	.75	.67
10-13	.94	.72	.66
13-11	.89	.75	.62
11-31	.80	.79	.69
31-39	.84	.79	.64
39-41	.85	.85	.62
41-43	.89	.72	.55
43-44	.69	.79	.52
44-38	.83	.78	.59
38-45	.86	.77	.58
45-46	.77	.88	.58
46-47	.90	.71	.55
47-40	.70	.72	.46
40-34	.57	.76	.36
34-42	.59	.44	.21

TABLE 8

Item Difficulty - Empirical Hierarchy (47 Tasks)

Task	Difficulty	Task	Difficulty
1 (Simplest)	1.00	14	.74
2	1.00	15	.75
3	.98	16	.70
4	.98	35	.72
6	.93	36	.71
7	.97	37	.70
8	.85	9	.69
17	.93	12	.53
18	.93	10	.72
19	.96	13	.54
20	.93	11	.69
21	.91	31	.62
22	.88	39	.61
23	.87	41	.62
24	.85	43	.48
29	.83	44	.58
30	.81	38	.57
32	.80	45	.55
33	.81	46	.61
5	.78	47	.48
25	.79	40	.48
26	.78	34	.35
27	.72	42 (Most Difficult)	.31
28	.75		

Discussion and Implications

An examination of the data in Table 7 (Walbesser ratios) indicated that with few exceptions task comparisons within the empirically determined hierarchy meet specific criteria on two of the three ratios. The consistency ratio was met in 34 of 46 comparisons, the adequacy ratio in 45 of 46, and the completeness ratio in 43 of 46. It should be noted that of the eleven comparisons not meeting the criteria level for the consistency ratio, five missed the criteria by .02 or less. The index of agreement for the hypothesized forty-seven task hierarchy was .63. After the tasks were rearranged in an empirical hierarchy based on Walbesser ratios the index of agreement was .75.

The empirically determined sequence is presented in Table 9. An examination of the sequence tends to support the following hunches for teaching youngsters the subtraction of rationals in fraction form when the difference is expressed in simplest form.

1. In subtracting rational numbers in fractional form, finding least common denominators is not as difficult for learners as the renaming(s) (sum and/or difference) required.

2. Within renaming categories I and II, tasks involving like denominators (3/4 - 2/4 = ___; 3 3/4 - 2 2/4 = ___; 3/4 - 3/4 = ___; 3 3/4 - 2 3/4 = ___) should be taught before proceding to tasks involving unlike denominators (1/2 - 1/3 = ___; 3 1/2 - 2 2/4 = ___; 4/8 - 2/4 = ___; 3 4/8 - 2/4 = ___).

3. Within renaming categories I and II, tasks involving unlike denominators (1/3 - 1/4 = ___; 4/8 - 2/4 = ___) should be taught before those involving like denominators in renaming category III (3/6 - 1/6 = ___).

4. Within renaming category III, tasks involving unlike denominators (5/12 - 1/6 = ___) should be taught before those involving like denominators in renaming categories IV and V (1 1/4 - 2/4 = ___; 1 1/6 - 3/6 = ___).

5. Within a renaming category, subtraction of two mixed numerals with whole number parts greater than one (6 1/6 - 3 2/4 = ___) should be taught before subtraction involving one mixed numeral (7 3/6 - 3/4 ___).

6. Tasks involving whole number sums greater than one (5 - 1/6 = ___) should be taught after work is completed with like denominators in renaming categories IV and V (1 1/4 - 2/4 = ___; 1 1/7 - 2/7 = ___; 1 1/6 - 3/6 = ___).

7. Tasks involving whole number sums (greater than one) requiring renaming in the difference (6 - 1 2/10 = ___; 5 2/4 = ___) should be taught before those which require no such renaming (3 - 1 1/4 = ___; 4 - 1/6 = ___).

TABLE 9
Empirical Hierarchy of Tasks

---

1. 3/4 - 2/4 = ___ (Simplest)
2. 2/3 - 1/3 = ___
3. 5/8 - 5/8 = ___
4. 2/5 - 2/5 = ___
6. 3 2/6 - 2 1/6 = ___
7. 3 4/7 - 2 3/7 = ___
8. 1 - 1/8 = ___
17. 1/4 - 1/8 = ___
18. 1/3 - 1/9 = ___
19. 1/2 - 1/3 = ___
20. 1/3 - 1/8 = ___
21. 1/6 - 1/8 = ___
22. 1/8 - 2/16 = ___
23. 4/14 - 2/7 = ___
24. 5/10 - 4/8 = ___
29. 3 1/8 - 2 1/6 = ___
30. 6 3/9 - 3 1/3 = ___
32. 4 1/6 - 1/12 = ___
33. 3 1/5 - 2/10 = ___
5. 3/6 - 1/6 = ___
25. 5/12 - 1/6 = ___
26. 2/5 - 2/10 = ___
27. 6/9 - 1/2 = ___
28. 3/4 - 3/6 = ___
14. 1 1/4 - 2/4 = ___
15. 1 1/7 - 2/7 = ___
16. 1 1/6 - 3/6 = ___
35. 1 1/8 - 1/4 = ___
36. 1 1/9 - 1/3 = ___
37. 1 1/5 - 2/3 = ___
9. 1 - 2/6 = ___
12. 3 - 1 1/4 = ___
10. 4 - 1/6 = ___
13. 6 - 1 2/10 = ___
11. 5 - 2/4 = ___
31. 8 4/8 - 2 1/3 = ___
39. 1 1/8 - 1/6 = ___
41. 1 1/5 - 4/10 = ___
43. 1 2/6 - 6/9 = ___
44. 4 1/5 - 2/7 = ___
38. 1 2/5 - 4/9 = ___
45. 6 1/6 - 3 2/4 = ___
46. 2 1/3 - 2/5 = ___
47. 7 3/6 - 3/4 = ___
40. 1 3/18 - 3/9 = ___
34. 5 3/5 - 3/9 = ___
42. 1 6/9 - 4/5 = ___ (Most Difficult)

8. Within renaming categories I, II, III the hypothesized ordering of unlike denominator classes (C, D, E, F, G) was verified. However no pattern emerges for sequencing denominator classes in renaming categories IV and V.

9. Sequencing of tasks in renaming categories IV and V is difficult. This may be a result of renaming interactions between sums and differences.

Caveat

In general the results of these studies, though limited to the addition and subtraction of rational numbers in fraction form, support the notion that both epistemological and psychological factors be considered when developing teaching sequences in mathematics. Some of the implications above would not necessarily be derived from logical analysis alone. Further, in these studies answers had to be expressed in simplest form. Other studies not requiring this condition might yield somewhat different results. (See addition hunches 3, 6 and subtraction hunches 1, 3, 4, 5, 6). Also, in interpreting the results of these studies one must be conscious of the limitations of indirect validation procedures. For example, confounding variables such as prior educational experience of subjects and errors of measurement must be considered.

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AN ORGANIZATIONAL MODEL OF ABILITIES RELATED TO INFERENCE IN MATHEMATICAL CONTEXTS

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Over a period of several years mathematics educators have been interested in the ability of teachers of elementary and secondary school mathematics to use propositional logic. Evidence of this interest abounds in several forms: many programs for preservice training of mathematics teachers require coursework on elementary logic; a large proportion of elementary teachers' mathematics and mathematics methods textbooks include chapters or units on logic. Logic is among the topics with which the Committee on the Undergraduate Program in Mathematics (1971) recommended elementary and secondary teachers of mathematics be conversant, and the National Council of Teachers of Mathematics, in its Guidelines for the Preparation of Teachers (1973) makes a similar recommendation. Articles such as the Exner-Hilton debate (Exner, 1971; Hilton, 1971) have appeared in journals devoted to the teaching of mathematics, and research on teachers' use of logical connectives (Gregory, 1972; Gregory & Osborne, 1975) and on teachers' ability to make inferences (Easterday & Henry, 1978; Eisenberg & McGinty, 1974; Jansson, 1975; Juraschek, 1978) has been reported in journals devoted to research on mathematics education.

On the other hand, one might argue that there is ample evidence in many of the same sources that while logic might be judged to be important it is not considered "all that important" by many mathematics educators. Although textbooks or programs may include chapters or courses on logic, these units are generally integrated poorly, if at all, with the remainder of the text materials; indeed textbooks on mathematics for elementary teachers frequently include logic as the last chapter. Eisenberg and McGinty (1975) have examined the contents of textbooks for elementary teachers to determine the coverage of logic and concluded that it was generally inadequate. The writer has examined one popular mathematics textbook for preservice elementary teachers (Kelley & Richert, 1970), coding all uses of logical connectives in the text and concluded that except for a few difficult arguments (e.g., the Fundamental Theorem of Arithmetic) understanding of disjunctive, conditional, or biconditional connectives or constructions is unnecessary to an understanding of the text content.

This finding is not entirely inconsistent with the statement of the Committee on the Undergraduate Program in Mathematics (CUPM):

> the elementary notions of logic such as logical connectives, negation, and the quantifiers should be treated explicitly only after attention has been called informally to their uses in other mathematical contexts. Indirect proofs and the use of counterexamples arise naturally and should be stressed when the

> structure of the number systems is examined. However, in the final stages of an elementary teacher's training it is useful to return to logic in a more explicit way (1971, p. 12).

However the ability of preservice elementary teachers to cope with indirect proof and the use of counterexamples in the absence of prior training on the use of propositional language and logic has been questioned, at least informally, by many teacher educators charged with teaching the structure of the number systems to preservice teachers.

Perhaps it is this nagging question which has led researchers such as Easterday and Henry (1978), Eisenberg and McGinty (1974), Jansson (1975), and Juraschek (1978) to examine performance of pre-service elementary teachers on tests of inference. The results have revealed poor performance on items involving disjunctive, conditional, and biconditional statements. However, the test items used by these researchers did not refer to mathematical concepts or arguments, but rather to familiar objects (cars, bikes, sisters, etc.) There would appear to be no reason to believe that training such as that recommended by CUPM should generalize to the colloquial contexts of these items; nor, conversely, is it obvious that performance on these items should transfer to mathematical contexts. Thus the importance of these studies to the concerns of CUPM and others is unclear. Moreover, even if the relevance of these studies is established, they provide little direction to the teacher educator.

Purpose and Assumptions

The purpose of the analyses and studies described below is to construct an organizational model, analogous to a learning hierarchy, which at once exhibits relationships among pre-service teachers' skills in the use and interpretation of logical connectives in mathematical contexts and provides direction to teachers of courses on mathematics and mathematics methods who seek to acquaint their students with those notions of logic recommended by CUPM.

Two assumptions underlie the development of the model explicated below. The first of these assumptions concerns the relationship between the context in which one is working (e.g., elementary mathematics) and the use of logical connectives; the second is concerned with the importance of the understanding of logic to elementary teachers.

The first assumption is an assertion that a principal factor determining the ability of individuals to use and interpret propositional language is the context in which the language is presented or requested. The context has at least three aspects: (1) the concepts or phenomena with which the statements deal, (2) the availability of additional information concerning these concepts or phenomena, and (3) the nature of the interpretive task. Usage which is "natural" or linguistically appropriate in one context might be "formal" or even artificial, as well as linguistically inappropriate, in other contexts, and vice-versa. Because there are fundamental relationships between logic and mathematics, the use of logic has an *a priori* importance in mathematical contexts. Moreover, the relationships between logic and mathematics lend a "natural" character to logical statements about mathematical concepts.

Examination of the research literature on the use of logic supports this assertion. Although the effects of mathematical versus nonmathematical contexts have not been studied sufficiently, several differences are apparent when one compares the results of studies in different contexts. The most striking difference concerns the disjunction. Juraschek (1978), using tests concerning nonmathematical concepts or objects, found strong evidence that subjects interpret the word "or" exclusively. In two studies using mathematical concepts (set membership, odd and even numbers), on the other hand, Damarin (1977a, 1977b) found that no subject ($N = 135$) consistently treated "or" as the exclusive disjunction. Rather, approximately one-third of the subjects interpreted "or" inclusively, while about half of them indicated that "p or q" was true only when both p and q were true; thus half the population treated the words "or" and "and" identically. The remaining one-sixth of the subjects were inconsistent in their interpretations of the word "or."

While the first assumption was related to the appropriateness and importance of studying the use of logical connectives in the context of elementary mathematics, the second addresses the relevance and importance of teachers' ability to use logical statements. The elementary school teacher is not concerned with the presentation of deductive arguments in the same sense as teachers of higher level mathematics. Rather, the teachers' use and interpretation of propositional statements lies in the communication of observations, generalizations, and rules, in the analysis of mathematical relationships underlying situations (as in story problems), and in the explanation of procedures, algorithms, and regularities. In these contexts, not only should teachers be able to draw valid inferences from clearly stated pairs (or strings) of statements; they also should be able to find counterexamples to fallacious statements, to present indirect arguments, and to translate natural language statements into other forms such as equations, inequalities, and lists. Conversely, they should be able to use appropriate compound logical statements to describe and interpret situations, categorizations, processes, rules, and mathematical relationships.

Definition of Terms

The discussion to follow concerns statements about relationships among mathematical variables. Each variable is defined over some domain; a statement concerning n variables is defined (though not necessarily true) over the Cartesian product of the n domains of the variables. For example, the statement "$xy>0$" is considered to be defined over $R \times R$ (R the real numbers), although it is certainly not uniformly true over $R \times R$. The domains over which the variables are defined are naturally partitioned by mathematical concepts or properties. R, for example, is naturally partitioned into sets of negative, zero, and positive numbers by the trichotomy property at zero. The positive integers (Z) are naturally partitioned by the concept "even;" another natural partition of the positive integers arises from the concept "prime."

The replacement set for a statement concerning a relationship among variables is the Cartesian product of sets of classes on the domains of the variables. The replacement set may be explicitly defined, or defined implicitly by the statement and its context. The statement "M + N is even" in the context of the question "Is M even?" implicitly defines the domain of each variable as Z (the positive integers), the domain of definition of the statement as Z x Z, and the classes within Z as "odd" and "even." The replacment set for this statement is, therefore [(M odd, N odd), (M odd, N even), (M even, N odd), (M even, N even).][1] The replacement set can be sorted into two subsets, the "truth set" consisting of all pairs of classes over which the statement is true, and its complement.

For any sorting of the replacement set into a truth set and its complement, there are infinitely many statements which correctly describe the sorting, and from which the sorting could be derived. The statements associated with a given sorting are said to be formally equivalent. Thus the statement "M + N is even" is formally equivalent to each of the following:

M and N have the same parity.
[(M+1) + (N+1)] is even.
M and N are both odd or both even.
M is odd if and only if N is odd.
M - N is even.

Although these statements are formally equivalent, they are not necessarily equivalent from the point of view of the facility with which they are interpreted or used. In order to investigate preservice elementary teachers' treatment of statements concerning relationships between variables, four types of statements were identified:[2]

1. Simple univariate statements: simple declarative statements concerning one variable, e.g., M is even, x>0, Angle A is acute.

2. Mathematical statements: simple declarative statements concerning the result of a mathematical operation on two or more variables, or in which two or more variables are related by a mathematical concept, e.g., M + N is even, xy>0, Angle A and Angle B are complementary.

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[1] In the tests used in the empirical work discussed below, the replacement was always given explicitly.

[2] The classification scheme is necessarily somewhat arbitrary due to the close relationship between mathematics and logic; the names for categories are even more arbitrary. The intent of the scheme was to differentiate, for the purposes of research, between statements in which relationships among quantities or variables are expressed by mathematical symbols or concepts and those in which these relationships are expressed by use of logical connectives.

3. Logical statements: Compound statements in one of the forms "p and q," "p or q," "if p, then q," or "p if and only if q," where p and q are simple univariate statements, e.g., M is even or N is odd. If the measure of Angle A is greater than 90, then the measure of Angle B is less than 90.

4. Complex logico-mathematical statements: Compound statements in which two or more logical or mathematical statements are joined, e.g., If M + N is even, then M - N is even.

The investigation focused on formally equivalent pairs of mathematical and logical statements concerning two variables. For such pairs of statements, there are three modes of presenting information concerning a relationship between two variables: a logical statement, a mathematical statement, or a sorting of the replacement set into a truth set and its complement. Table 1 gives examples of presentation of the same information in each of three modes.

Table 1

Presentation of Equivalent Information in 3 Modes

Mathematical Statement	Logical Statement	Sorting of the Replacement Set
M + N is odd	M is even if and only if N is odd	Truth set: [(M odd, N even), (M even, N odd)] Complement: [(M odd, N odd), (M even, N even)]
M x N is even	M is even or N is even (or) If M is odd, then N is even	Truth set: [(M odd, N even), (M even, N odd), (M even, N even)] Complement: [(M odd, N odd)]
M x N is odd	M is odd and N is odd	Truth set: [(M odd, N odd)
$xy = 0$	$x=0$ or $y=0$ (or) If $x \neq 0$, then $y=0$	Truth set: [($x=0,y=0$), ($x=0,y\neq 0$), ($x\neq 0,y=0$)]
$ab>0$	$a>0$ if and only if $b>0$	Truth set: [($a>0,b>0$), ($a<0,b<0$)]

Any piece of information which can be presented in one of these modes can also be stated in either of the others; translation from one mode to another is frequently a critical component of indirect proof and the identification of counterexamples. There are six translation tasks which can be performed among the three modes; these are identified and named in Table 2.

Table 2

Descriptions of Six Translation Tasks

Task Name	Translation Task
L→R	Given a logical statement, sort the set of possibilities by indicating for which possibilities the statement is true.
M→R	Given a mathematical statement, sort the set of possibilities by indicating for which possibilities the statement is true.
L→M	Given a logical statement select an equivalent mathematical statement.
M→L	Given a mathematical statement select an equivalent logical statement.
R→L	Given a sorting of the set of possibilities select a logical statement describing the truth set.
R→M	Given a sorting of the set of possibilities select a mathematical statement describing the truth set.

Questions concerning the relationships both among these tasks and between the inference process and these tasks can be posed at many levels. Are these translations constituent parts of the inference process? Do some translations occur as parts of others? For example, does performance of translation L→M consist of performing first L→R and then R→M? Can individuals who can perform one translation also perform its reverse? Can translation L→M be performed by persons who cannot perform translation L→R? Numerous other questions might be addressed.

It is not the purpose here to address the question of whether the translation processes outlined above are actually involved in the cognitive processing of statements and the extraction of inferences from them. Rather the point of view adopted is that individuals draw inferences on the bases of their understandings of the meanings of statements, and that the depth of understanding is related to the ability to translate the statement to equivalent statements in the same mode or in other modes. Moreover, systematic errors in inference can be explained by misunderstandings which are revealed in systematic translation errors. The model described below is therefore posed, not as a potential model for the actual cognitive processing of statements (though it may prove to be connected to it), but rather as a model of the organization of skills related to the understanding of, translation of, and inference from statements about mathematical concepts and variables.

The Model

The understanding of mathematical argument, proof, and counterexample seems to involve the understanding of mathematical statements, understanding of logical statements, and the ability to make translations and comparisons back and forth between logical and mathematical modes.

The understanding of statements in each mode has two aspects: interpretive understanding and constructive understanding. Interpretive understanding of a given statement is operationally defined as the ability to partition the corresponding replacement set into the truth set for the statement and its complement. Thus, the making of a truth table for a statement would evidence interpretive understanding of that statement. Less formal evidence is more likely to appear in the classroom; for example, the observation "they both have the same sign" would evidence interpretive understanding of the statement "$xy>0$."

Constructive understanding, on the other hand, is the ability to use a logical or mathematical statement appropriately in describing a sorting the replacement set. For example, consider the set of whole numbers as the replacement set; the sequence 3, 5, 6, 9, 10, 12, 18, 20, 21, 24, 25, 27, 30, 33, 35, 36,....partitions the whole numbers into a sequence (truth set) and its complement. An individual who describes the sequence as the set of whole numbers which are "multiples of 3 or multiples of 5" is illustrating constructive understanding of the (inclusive) or.

The ability to make translations L→R or M→R listed in Table 2 is indicative of interpretive understanding (of logical or mathematical statements, respectively), while the ability to make translations R→L or R→M indicates constructive understanding.

Translation from a mathematical statement to a logical statement would appear to require interpretation of the mathematical statement and then construction of the logical statement from the interpretation. An analogous pair of operations is involved in translation from logical to a mathematical statement. M→L translation from a mathematical to an equivalent logical statement is therefore modelled as the composite of skills M→R and R→L. Similarly L→M is modelled as the composite of skills L→M and R→M.

Figure 1 below illustrates the model. In the absence of evidence establishing relationships between or among L→R, R→L, M→R, and R→M, these skills are considered to be separate. On the basis of the a priori analysis above L→M is considered as the composition of translations L→R and R→M; similarly M→R is composed from M→R and R→L.

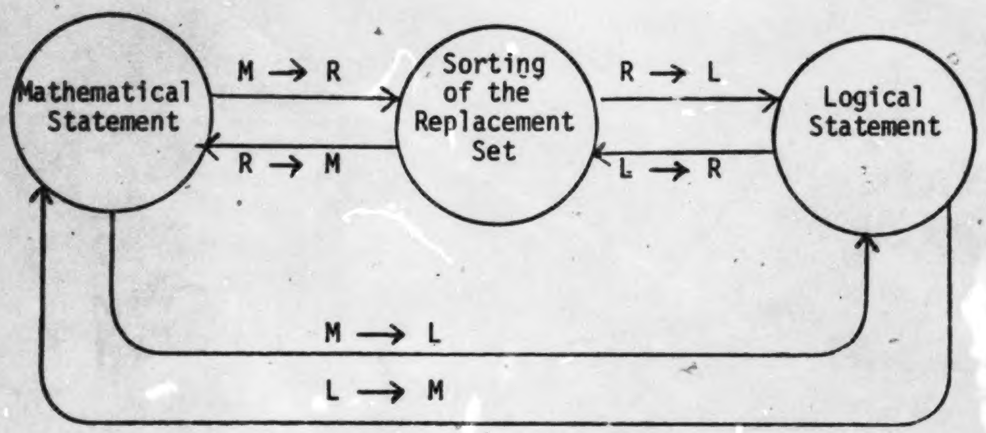

Figure 1. The organization model.

Empirical Studies: Support for the Model

A number of investigations related to this model have been conducted using populations of preservice elementary teachers (Damarin 1976, 1977a, 1977b, 1978, in press). These studies have shown that L→R and R→L, interpretive and constructive understanding of logical statements about mathematical variables, are distinct abilities with constructive understanding being more prevalent among the population than interpretive understanding (Damarin, in press). In addition, translation from mathematical to logical statements is easier for the population than the converse translations. Interpretive and constructive understanding of the mathematical statements tested were correlated.

The hypothesis that the ability to translate from a mathematical to a logical statement is dependent upon the interpretive understanding of the mathematical statement and the constructive understanding of the logical statement was supported. However, the low scores on both tests of interpretative understanding of logical statements and translation from logical to mathematical statements precluded drawing any conclusion except that these skills are rare in the population.

Empirical Studies: Understanding logical statements. Two studies (1977a, 1977b) revealed that preservice elementary teachers interpret both conditional and biconditional statements about mathematical concepts in the same way they interpret conjunctions; that is, the only member of the replacement set assigned to the truth set is the element satisfying both simple statements. The same studies showed that slightly less than one-third of the preservice elementary teachers tested interpreted the connective "or" as an inclusive disjunction, and more than forty percent interpreted it as a conjunction (the remainder were inconsistent in their responses). Response patterns on tests of constructive understanding were observed to be similar but less well-defined (1978).

Because evidence concerning interpretation of "or" stands in stark contrast to the findings of other researchers (Eisenberg & McGinty, 1974; Jansson, 1975; Juraschek, 1978) who used inference tests in non-mathematical contexts, an attempt was made to determine whether the context of statements determined the interpretive understanding of the word "or" (Damarin, in press). Inference items and interpretive understanding items were constructed in mathematical, technical-scientific, and familiar content areas yielding six tests; tests were ordered randomly for administration to preservice elementary teachers. These subjects were more likely to interpret "or" as an inclusive disjunction or conjunction when statements dealt with mathematics than when statements were drawn from the other contexts. An order effect was apparent in the data; after some inconsistency on the first several items many subjects tended to settle on a single interpretation of "or" and use it on the remaining items regardless of context.

The Model in a Research Context

This simple model has rather obvious relationships to several lines of educational and psychological research, as well as to the truth tables of elementary logic.

Beilen (1976) pointed out that learning of mathematical concepts or relations is a complex function of two variables: the nature of the mathematics to be learned and the psychology of the learner. Models related to learning processes can be developed from points of view that are essentially mathematical, essentially psychological, or reliant upon both mathematics and psychology.

Because the separation of logic and psychology has been a significant problem in the history of both fields (Bourne, Ekstrand, & Dominowski, 1971; Beth & Piaget, 1966), it is important to delineate fairly carefully the relationship of each discipline to the model. The foundation of the model is the truth table of elementary logic. Using truth tables as a guide it is possible to establish a formal method of determining equivalences among certain classes of logical amd mathematical statements. This formal equivalence (as detailed fully in Damarin, 1976) is defined in relation to the replacement set; two statements are formally equivalent provided they are associated with the same sorting. The definitions of statement equivalence and sortings are thus derived from logic and mathematics.

On the other hand, both the methods an individual uses in finding a sorting or for learning to sort the set of possibilities appropriately for a given problem situation, and the methods he uses in assigning a statement to a given sorting are psychological problems. These methods seem likely to be related to the process of combinatorial thought discussed by Piaget (1957; Inhelder &Piaget, 1958; Beth & Piaget, 1966).

Although Piaget does not deal directly with the use of propositional statements in mathematical contexts, this theory can be interpreted as relating the three presentation modes. Underlying Piagetian discussions of logical thinking or combinatorial thought is a three stage model for the process of solving problems by logical methods. During the first stage the problem is presented. The second stage consists of the systematic hypthosizing and testing of possible solutions or partial solutions. During this stage the INRC group operates, according to Piaget, to monitor the order in which possibilities are tested. Finally, the problem solution is expressed in a logical statement. This model is illustrated in Figure 2.

Piaget's model was developed with reference to problems drawn primarily from the empirical sciences and has been subject to test only with problems in that domain. In these problems consideration and testing of possibilities can be viewed as a unitary process because the actual testing of possibilities considered is a trivial matter. Once a subject has determined to combine three colorless liquids, for example, the actual test requires only the simple act of pouring. The consideration and testing procedure necessary for solution of problems involving mathematical relationships between variables is frequently more complex than it is in Piaget's experimental tasks. After an individual determines an element of the replacement set, his test of that possibility will generally require the use of an algorithm which he may or may not execute correctly. If the algorithm is performed incorrectly the subject may erroneously accept or reject the possibility under consideration. Moreover, the performance of the algorithm, or its results, may suggest other potential solutions (e.g., extraneous roots) which could be helpful or misleading. Thus the M→R translation, interpretive understanding of mathematical statements, should be considered as consisting of two parts, generation of the replacement set and

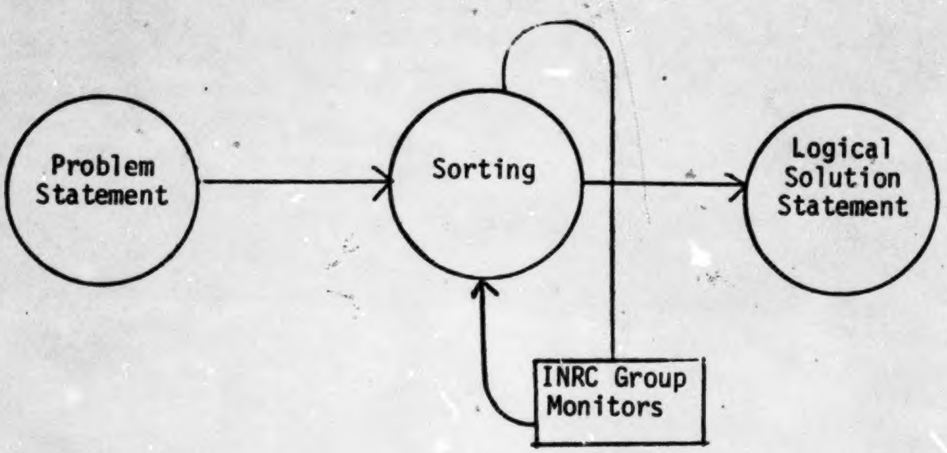

Figure 2. The Piagetian Model.

sorting the set using algorithms. The first of these processes is addressed by Piaget in discussions of the INRC group (Piaget, 1957; Inhelder & Piaget, 1958).

A similar issue necessitates the consideration of translation L (interpretive understanding of logical statements) as potentially consisting of two parts also. In discussing subjects' performance of inference tasks, Wason and Johnson-Laird (1970) observe that in many cases when subjects are given "if...then..." statements they simply do not consider the possibility that the antecedent is false. That is, they do not generate the elements (p false, q false) and (p false, q true) as members of the replacement set for "if p then q." However, the evidence that constructive understanding of logical statements is easier than interpretive understanding suggests that, were these possibilities generated, they might be evaluated correctly. Therefore interpretive understanding of logical statements must be viewed (tentatively, at least) as involving two processes: generation of the replacement set and evaluating the elements of the set generated.

In light of these considerations the model can be viewed in an expanded form as illustrated in Figure 3. The expanded model facilitates the explication of at least theoretical relationships between a variety of research findings and the simple model and, thus, the formulation of research questions and hypotheses.

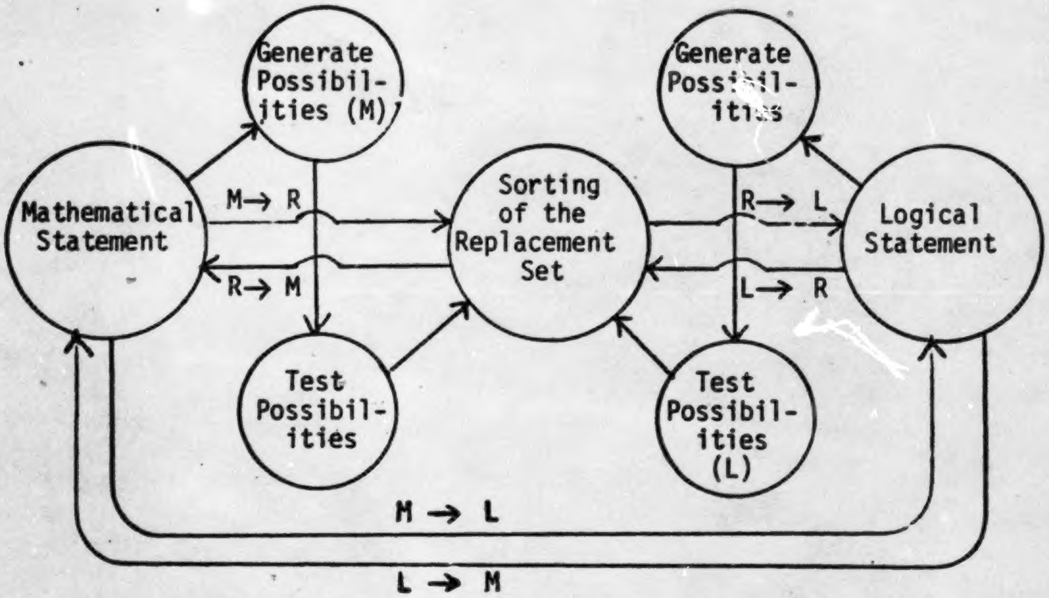
Generate Possibilities (M)
Generate Possibilities
Mathematical Statement
M→ R
R→ M
Sorting of the Replacement Set
R→ L
L→ R
Logical Statement
Test Possibilities
Test Possibilities (L)
M → L
L → M

Figure 3. The expanded model.

One example of using the model to integrate existing literature deals with implication. Shapiro and O'Brien (1973) identified as "Child Logic" the common tendency of their subjects to respond to conditional inference items as if they were biconditional; "Child Logic" is then considered as the failure to distinguish conditional and biconditional statements. Damarin (1977a, 1977b) reports that (in mathematical contexts) both conditional and biconditional statements are interpreted as conjunctions, and other researchers (e.g., Wason & Johnson-Laird, 1970) report that subjects do not always spontaneously consider cases in which p is false when interpreting statements of the form "if p then q."

By isolating the generation of the replacement set, the model affords the possibility of reinterpreting the results of Shapiro and O'Brien (1973) and Damarin (1977) and finding them consistent. Under the assumption that subjects are generating and sorting only the partial replacement set (p true, q true), (p true, q false) the response patterns to inference items would be exactly those associated with correct processing of biconditional statements. These patterns are illustrated in Table 3.

Table 3. Response patterns for inference items

Given	Question	Conditional Interpretation Response	Biconditional Interpretation Response	Restricted Replacement Response
$p \supset q, p$	q?	Yes	yes	yes
$p \supset q, q$	p?	can't tell	yes	yes
$p \supset q, \bar{p}$	q?	can't tell	no	impossible (no)
$p \supset q, \bar{q}$	p?	no	no	$p\bar{q}$ in false set (no)

Designing Curricula

Both the model and the expanded model have as their cores the partition of the replacement set into a truth set and its complement. Thus the primary "message" of the model to curriculum developers is the central position of the replacement set and its sorting in the ability to translate back and forth between mathematical and logical statements.

One study inspired by the model and supporting the importance of the replacement set to teachers' analyses of mathematical arguments has been reported (Damarin, 1977c). In this study preservice elementary teachers were given a series of arguments some valid and some not, purported to be parts of elementary students' discussions of standardized test problems. The preservice teachers were asked both to judge the validity of the arguments and to provide students with feedback on invalid arguments. Half of the teachers were given the replacement set as a set of options for feedback (in the form of counterexamples), while the other half were required to write their own feedback. While no difference between the groups was found for the number of correct judgements of arguments as valid or invalid, there was a significant difference between the groups in the types of errors made. Those who were given the replacement sets tended to err by rejecting valid arguments, while those not given the set erred by accepting invalid arguments.

Without condoning errors of either type, a strong argument can be made that it is better for teachers to reject the valid arguments of students than to accept their invalid arguments. In the former case the student can seek clarification, make the argument again, or reject the teacher's judgement; in the latter case the student's misunderstandings are reinforced.

This study, in conjunction with the model, suggests strongly to curriculum developers that exercises in the generation and sorting of the replacement sets for statements and arguments would be of value to preservice elementary teachers. In particular, this generation and sorting might be made an integral part of the study of "indirect proofs and the use of counterexamples which arise naturally and should be stressed when the structure of the number systems is examined (CUPM, 1971, p. 12)."

A second implication of the model for teacher education curricula is the need to distinguish between interpretive and constructive understanding of logical statements. Research concerning this model indicates that constructive understanding of logical statements is much more prevalent among preservice elementary teachers than interpretive understanding. This finding suggests that a more effective approach to explicit work on logic than the common truth tables (interpretive) introduction could be achieved by first solidifying constructive understanding through work on description of set partitions. Subsequently, truth tables could be introduced as isomorphs of set partitions and their use examined.

Research designed to test the model revealed that in the mathematical contexts examined the word "or" was interpreted appropriately by one-third of the preservice teachers tested; moreover none of these subjects interpreted "or" exclusively in these contexts. Very few subjects treated conditional statements correctly. These findings suggest that arguments (and statements for which counterexamples are to be found) be stated initially in disjunctive form. After the analysis of disjunctive arguments is mastered, conditional statements should be related to the equivalent disjunctions.

Finally, many, if not all, texts which include sections on logic use statements about familiar objects almost exclusively in their presentations of logic to preservice teachers. Investigation of this model and the subsequent study (Damarin, in press) indicate that preservice teachers use and interpret the word "or" differently in mathematical and nonmathematical contexts. Textbook use of familiar situations, while intended to provide the novice student with a more "natural" context than mathematics in which to reason, may actually impede learning by confounding technical and colloquial usages of the connective words. An approach to the explicit teaching of logic wholly within the context of mathematics would avoid this confounding.

Conclusion

The model examined is not derived from any overarching learning theory or theory of mathematics education, but rather by addressing the pragmatic concerns of the mathematics educator who is trying to teach preservice elementary teachers to reason logically in mathematical contexts. A teaching process which attains this goal serves as the prototype for the model, although this process currently exists only in the abstract. This process must take into account the subject matter being taught and relevant aspects of the nature of the preservice teacher as revealed by research on use of logic, combinatorial reasoning, and other cognitive processes.

Richards (1979) in his discussion of models in mathematics education, states "(a model) is not intended to capture totally or to explain a situation. A model simplifies, represents, visualizes, preserves structure (mainly), and finally disappears." The organizational model described here preserves the structure of the subject matter, the structure of the curricular objectives and emphases recommended by CUPM, and the known structural relationships among relevant cognitive skills of preservice elementary teachers. In its expanded form the model affords synthesis and interpretation of several lines of research (e.g., Piagetian, relevant areas of experimental psychology) related to the teaching problem, and thus simplifies the relationship of the literature to the teaching problem. The model provides a visual display of the relationships among discrete processes or skills identified as components of the ability to use logic in the mathematical contexts (at an appropriate level). Insofar as the model is used in the development of curricula, the model will also disappear, being replaced, not by a theory, but by the curriculum itself or by other models which are responsive to other aspects of the curricular and instructional problem.

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LOGICAL REASONING LEARNING HIERARCHIES

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Piaget (Inhelder & Piaget, 1958) has suggested that formal logical reasoning can be described in terms of sixteen binary combinations of the four possible associations of simple propositions and their negations: p, $\bar{p}$, q, and $\bar{q}$. In the work of the Geneva group, no developmental or learning sequence has been indicated for these sixteen logical operations (see Table 1). Airasian, Bart, and Greaney (1975) have used ordering theory to develop a hierarchy among the sixteen operations. It was the aim of the present study to:

1. Refine some of the procedures and definitions required to utilize ordering theory,
2. Replicate the earlier study of Airasian, Bart, & Greaney, and
3. Extend this type of investigation to a broader age range.

Furthermore, it was hypothesized that if the ordering theoretic approach is a valid and reliable one, the hierarchies found would not differ significantly by age level. This hypothesis, of course, makes the assumption that the developmental sequence for logical operations is essentially invariant over individuals. A finding of similar hierarchies would thus suggest that 1) the approach, in both its rationale and its application, was appropriate, and 2) logical operations do indeed develop in a fixed sequence. On the other hand, a finding of significantly different hierarchies would indicate the negation of the first and/or second suggestions.

The study of the relationship of logic to mathematics and to intellectual development has a rich, if not long, history. Clearly the three are interrelated and particularly the relationship of logical reasoning to intellectual development has been the subject of recent educational and psychological research (e.g., Wason & Johnson-Laird, 1968; Bart, 1970, note 1; Hooper, Brainerd, & Sipple, 1975; Lunzer, 1975, note 2; Blake, Lawson & Nordlund, 1976; Shayer, 1978, note 3; Sigel & Hooper, 1969), much of it based upon the work of Piaget and his collaborators (esp. Inhelder & Piaget, 1958). While the logic of Piaget has been severely criticized by Ennis (1975, 1976), empirical evidence (e.g., Jansson, 1977, 1978; O'Brien, 1972; Roberge, 1970) does suggest that logical reasoning develops with chronological maturation. To the mathematics educator and curriculum developer, an understanding of this development is critical, for the very nature of mathematics is inherently tied to logic.

The paper which follows looks at ordering theory as a tool for the construction of a model of the development of logical reasoning. The main sections will give a brief summary of ordering theory, an explanation of the procedures used in the empirical investigation, and the results of the investigation.

Table 1

Piaget's System of Sixteen Binary Operations

Piaget's Name	KBA Name	Constructed Combination
complete affirmation	(not used in study)	$p.q \vee p.\bar{q} \vee \bar{p}.q \vee \bar{p}.\bar{q}$
complete negation	(not used in study)	nothing
conjunction	conjunction	$p.q$
incompatibility	incompatibility	$p.\bar{q} \vee \bar{p}.q \vee \bar{p}.\bar{q}$
disjunction	non-exclusive disjunction	$p.q \vee p.\bar{q} \vee \bar{p}.q$
conjunctive negation	conjunctive negation	$\bar{p}.\bar{q}$
implication	implication	$p.q \vee \bar{p}.q \vee \bar{p}.\bar{q}$
non-implication	non-implication	$p.\bar{q}$
reciprocal implication	inverse implication	$p.q \vee p.\bar{q} \vee \bar{p}.\bar{q}$
neg. of reciprocal implication	inverse non-implication	$\bar{p}.q$
equivalence	equivalence	$p.q \vee \bar{p}.\bar{q}$
reciprocal exclusion	exclusion	$p.\bar{q} \vee \bar{p}.q$
affirmation of p	affirmation of p	$p.q \vee p.\bar{q}$
negation of p	negation of p	$\bar{p}.q \vee \bar{p}.\bar{q}$
affirmation of q	affirmation of q	$p.q \vee \bar{p}.q$
negation of q	negation of q	$p.\bar{q} \vee \bar{p}.\bar{q}$

Ordering Theory

In a developmental sense ordering theory has its foundations in the work of Guttman and others who developed linear scaling techniques. Recent developments in scaling models now permit nonlinear, or branching, trees to be constructed from a set of items. The usefulness of such techniques in building learning hierarchies is obvious. The brief statement here is based upon the work of Krus, Bart, and Airasian (1975).

Definition

The basic notion of scaling is that a set of bivalued items having a common skill, notion, or theme may be ordered in the sense of "item i is pre-requisite to item j." Krus, et al. (1975) present evidence that not all orderings among items are linear. An oversimplified description of the approach and an example follow.

Let "1" represent a correct response, and "0" an incorrect response to an item. Then the prerequisite relation may be defined as follows:

> "Success on item _i_ is a prerequisite to success on item _j_ if and only if the response pattern (0,1) for items _i_ and _j_ respectively does not occur. The response patterns (0,0) (1,0), and (1,1) are called _confirmatory_ and the response pattern (0,1) is called _disconfirmatory_ with respect to the two-item ordering. [Thus] for a set of _n_ items such that a correct response for any item indicates a manifestation of the trait being measured one can construct a matrix which indicates the percentage of disconfirmatory response patterns for all of the two-item trees possible among the _n_ items. In other words, an _n_ x _n_ matrix, indexed along the rows and columns by the item numbers, may be constructed such that the entry in the cell in the _i_th row and _j_th column is the percentage of the total response patterns that had an "0" for the _i_th item and an "1" for the _j_th column. With such a table one may consider a given tolerance level of disconfirmatory response patterns and then identify item pairs that are related by a prerequisite relationship. Thus if a cell entry for row 2 and column 5 in the matrix is .2 and one pre-established a tolerance level of 1 percent disconfirmatory response patterns, then since .2 percent is within the 1 percent tolerance level one can say that item 2 is prerequisite to item 5 " (Krus, Bart, & Airasian, 1975, p. 48).

The tolerance level referred to above establishes the proportion of disconfirmatory response patterns which will be accepted in defining a prerequisite relationship between two items. This technique is used to overcome the lack of a method for dealing with the possibility of random error in item response patterns which is inherent in the deterministic measurement model.

Suppose that we have three items, A, B, and C. Since the items are bivalued, we have eight possible response patterns. We could, for example, hypothesize an ordering such as that shown in Figure 1. Also indicated are the confirmatory and disconfirmatory response patterns. On the other hand, if we wish to develop a hierarchy from scratch or verify a given hierarchy we would look at the frequency with which each response pattern occured. A 3x3 matrix of percentages of disconfirmatory patterns for item pairs would permit us to construct an ordering, displayable in graphical or tree form. For the three items in our example we might obtain something like that shown in Table 2. For instance, row B column C shows 72% disconfirmatory response patterns, hence B is _not_ prerequisite to item C. On the other hand, row C column B indicates no disconfirmatory response patterns, therefore C is prerequisite to B.

A	B	C	Values
0	0	0	C
0	0	1	C
0	1	0	D
0	1	1	D
1	0	0	C
1	0	1	C
1	1	0	D
1	1	1	C

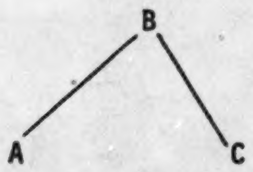


Figure 1. Response Patterns for an hypothesized three-item hierarchy.

Table 2

Matrix of Percentages of Disconfirmatory Response Patterns for Two-Item Prerequisite Relation Orderings among the Three Items in the Example

Items	A	B	C
A	--	0	0
B	60	--	72
C	0	0	--

Thus there are two types of tasks which might be accomplished by ordering theory: 1) to test a hypothesized ordering among items, and 2) to empirically generate an ordering among items when no order is hypothesized. After various prerequisite-related item pairs are identified, one may construct a hierarchy among the items. Different item hierarchies will be constructed for different tolerance levels. The selection of a low tolerance level makes the method closer to the logical definition of the prerequisite relationship.

A careful reading of the description of the measurement model makes it clear that the relationship between two items may be other than prerequisite. In order to further refine and explicate the theory some notation will be introduced. Basically, the focus of the data analysis is the 2x2 table in which the response patterns for an item pair are evident. Thus for the item pair (*i*, *j*), we have a 2x2 table with "0" and "1" rows for *i* and "0" and "1" columns for *j*. Let P(0,1) be the proportion of responses in cell (0,1). Then the basic definition states:

> If $P(0,1) \leq t$ for items *i* and *j* respectively, then the relationship *i prerequisite to j* is confirmed for tolerance level *t*.

As Eisenberg (1976) points out, however, this is an insufficent algorithm. In fact, it can occur that both $P(0,1) < t$ and $P(1,0) < t$. Furthermore, there are two other between-item relationships possible as Airasian, Bart, and Greaney (1975) describe without adequate definition. They state that "an item is logically equivalent to another item if success on one item cooccurs with success on the other item" and that two items are logically independent "if the frequencies of the four possible response patterns occuring between two items . . . all exceed the tolerance level" (p. 21). These characterizations do not provide a complete, well-defined partition of the possible outcomes.

The description of equivalence suggests that two items *i* and *j* would be equivalent if $P(0,0)$ and/or $P(1,1)$ were large and $P(0,1)$ and $P(1,0)$ were consequently small. Independence, to be distinguishable from equivalence, would appropriately have both $P(0,1)$ and $P(1,0)$ large. The occurance of large numbers of (0,0) or (1,1) response patterns neither confirms nor disconfirms independence. Given these considerations, we can define a set of mutually exclusive relationships for any item pair based upon the 2x2 table. Let the tolerance level be *t*. Then for any item pair (*i*, *j*):

criteria	relationship
$P(0,1) \leq t$ and $P(1,0) \geq t$	*i* prerequisite to *j*
$P(0,1) \leq t$ and $P(1,0) < t$	*i* equivalent to *j*
$P(0,1) > t$ and $P(1,0) \geq t$	*i* independent of *j*
$P(0,1) > t$ and $P(1,0) < t$	*j* prerequisite to *i*

This definition provides a partition of the possible $P(i, j)$ relationships to *t*, and hence a decision making algorithm for construction of the hierarchy. Note that $P(0,1)$ for $(i, j) = P(1,0)$ for (j, i), hence the last line could be omitted for a more parsimonious definition. Furthermore, while equivalence possesses all three properties expected of an equivalence relation, the prerequisite relation is transitive (within tolerance levels) without being symmetric or reflexive. Independence is symmetric, but not reflexive nor necessarily transitive. From a theoretical point of view, we would like to have a stringent tolerance level *t*. Because of the exploratory nature of the empirical investigation to be described and the small sample sizes, a tolerance level of $t = .10$ was selected. In practice, since it is the (0,1) and (1,0) cells which are critical, an *nxn* table such as Table 2 is constructed. This table is readily interpretable if the items for row and column are ordered by difficulty level.

Discussion

The above definition retains the basic notion of the prerequisite relationship given by Krus, et al. (1975) and operationalizes the other complementary relationships. While one might argue with certain aspects of ordering theory as presented, it was the aim of this paper to operationalize and then apply the theory rather than to present a critical evaluation of it.

A number of other writers have discussed learning hierarchy construction from a quantitative viewpoint. In particular, White (1973, 1974) has provided a model for learning hierarchy research and reviewed the literature. More recently, Cotton, Gallagher, and Marshall (1977) have presented a mathematical model for examining intra-hierarchy vertical transfer among subskills. They suggest that skill on an item may simply transfer to the next higher item rather than be fully prerequisite to it. In other cases two vertically paired items may be learned at the same time. Most of the literature and previous investigations, including Uprichard and Phillips (1977, 1979) in mathematics education, postulate a hierarchy and teaching of specific skills to criterion. While the approach employed in this paper may be used to verify a hypothesized hierarchy, it also may be used to create a hierarchy where none has been proposed a priori. An assumption of some relatedness of the tasks must exist, however, for although totally unrelated tasks would yield analyzable data, they would not imply true hierarchical relationships.

A Replication Study

Airasian, Bart, & Greaney (1975) presented an ordering theoretic analysis of a propositional logic game. Given the important relationship between logic and mathematics, a major aim of this investigation was to replicate that earlier study.

Instrumentation

The propositional logic game, "Butch and Slim," has been developed as part of the new British Intelligence Scale (Ward & Fitzpatrick, 1970) and is based upon Piaget's sixteen binary combinations, with the exclusion of complete affirmation and complete negation. In place of these, two other verbal statements corresponding to non-exclusive disjunction and incompatibility are included. In each case one statement is in conditional form and the other is not. It is "played" by a single subject and an interviewer. Butch and Slim are two bank robbers who are questioned (together) by the police. Subjects have in front of them a set of four picture cards containing statements and pictures of the robbers (Ward, 1972, pp. 268-270). The following instructions were given to the subject:

> This is a game about a robbery. There has been a robbery and two well-known criminals, 'Butch' and 'Slim' are suspected. Here is Butch (point), and here is Slim (point). Now, supposing the two were taken in and questioned together, in the same room, as to whether they committed the crime, they could really answer in only four ways: both of them could say 'yes' they did it, like this (interviewer points to card 1) or Butch could say that he did it and Slim deny it, like this (point to card 2). Or Butch might deny that he did it, but Slim admit it, like this (point to card 3). Then, of course, they might

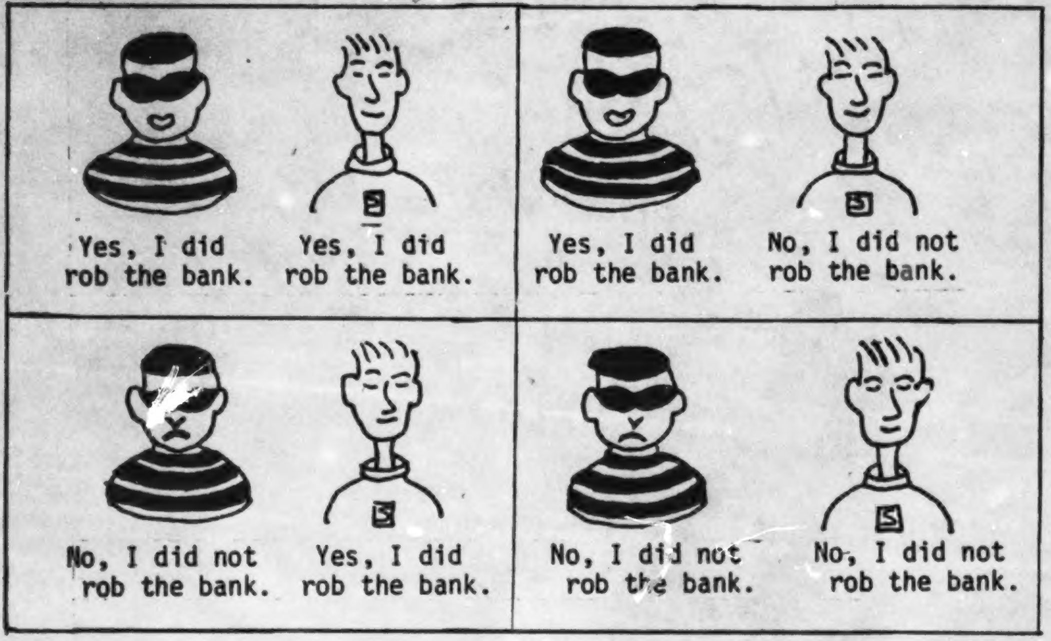

Yes, I did rob the bank.
Yes, I did rob the bank.
Yes, I did rob the bank.
No, I did not rob the bank.
No, I did not rob the bank.
Yes, I did rob the bank.
No, I did not rob the bank.
No, I did not rob the bank.

Figure 2. Cards for "Butch and Slim"

> both say 'no', like this (point to card 4). So when they were asked if they committed the crime, they could answer in the four ways shown on these cards. Now I am going to tell you some statements Butch might have made to the police, and each time I want you to look at each of these cards in turn and tell me whether Butch and Slim's answers on the cards could be true or not. You are looking for cards which could agree with what Butch said (Based upon Ward, 1972, p. 269).

A sampling of Butch's statements is presented in Table 3. Most of the sixteen items require the selection of more than one of the cards. Hints to subjects were provided as outlined by Ward (1972). Specifically, these provided admonitions to look for all possible cards following items 1, 3, and 12. This was necessary to preclude the subjects viewing the situation as a multiple choice one.

Subjects

A sample of 50 grade nine subjects participated in this study. They were selected randomly, from mathematics and science classes in a Winnipeg junior high school. While the Airasian, Bart, and Greaney (1975) sample consisted exclusively of girls, this one was mixed but sex was not considered as a variable in the analysis because earlier research (e.g., O'Brien & Shapiro, 1968) has identified no sex differences in formal operational thinking. The subjects had had no prior explicit training in logical reasoning.

Table 3

Sample "Butch & Slim" Items

Q. No.	Butch's Statement	Cards Correct	Description
3	I am certain Slim did not rob the bank.	2,4	Negation of q $(p.\bar{q})v(\bar{p}.\bar{q})$
10	Only one of us robbed the bank.	2,3	Exclusion $(p.\bar{q})v(\bar{p}.q)$
12	If we did rob the bank we did it together.	1,4	Equivalence $(p.q)v(\bar{p}.\bar{q})$
14	If I robbed the bank, Slim robbed the bank.	1,3,4	Implication $(p.q)v(\bar{p}.q)v(\bar{p}.\bar{q})$

Procedures

The game was administered to each of the 50 subjects and items were scored dichotomously. A subject had to select the exact set of pictures comprising a correct response to be given a score of "1" on the item. No credit was given for incomplete or incorrect responses, which were scored "0." The data then were analyzed using the ordering theoretic method described above. A tolerance level of .10 was used to conform to the earlier study. This seems appropriate in view of the relatively small sample size, the exploratory nature of the investigation, and the potential for guessing the correct response.

Results

Since the principal aim of this phase was the replication of the earlier study, the logical ordering hierarchy obtained by that research is presented in Figure 3. The sample consisted of subjects 14 years of age. The logical ordering resulting from this sample is given in Figure 4. In these and later figures, for the two duplicated operations the conditional is designated by a double prime (") and the nonconditional by a single prime ('). The double arrows indicate equivalence, while no connecting line between two elements shows independence. In Figure 4, for example, affirmation of p and affirmation of q are equivalent, but both are prerequisite to non-exclusive disjunction.

At first glance these two trees appear quite dissimilar. With the exception, however, of equivalence, and to some degree of negation of q, there is much alike in the two figures. The four operations at the top are equivalent and all were answered correctly by less than 4% of the subjects. For the replication study the items in the box at the bottom of the hierarchy are given in order of difficulty, i.e., the bottom row is easier than the top row and the items in each row were more difficult from left to right. The items in the box were responded to correctly by at least 90% of the subjects. Looking at the items in the hierarchy from bottom to top,

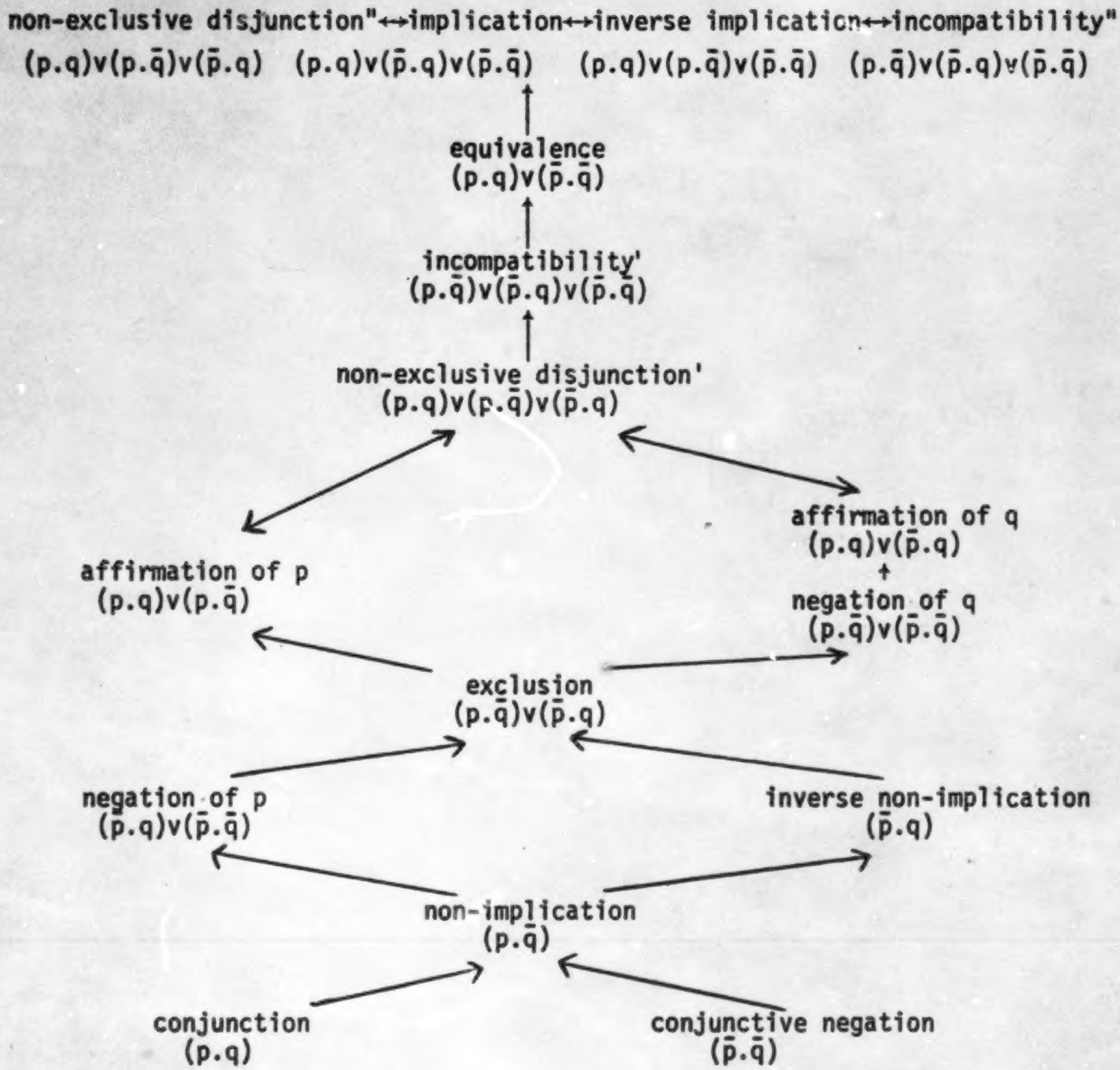
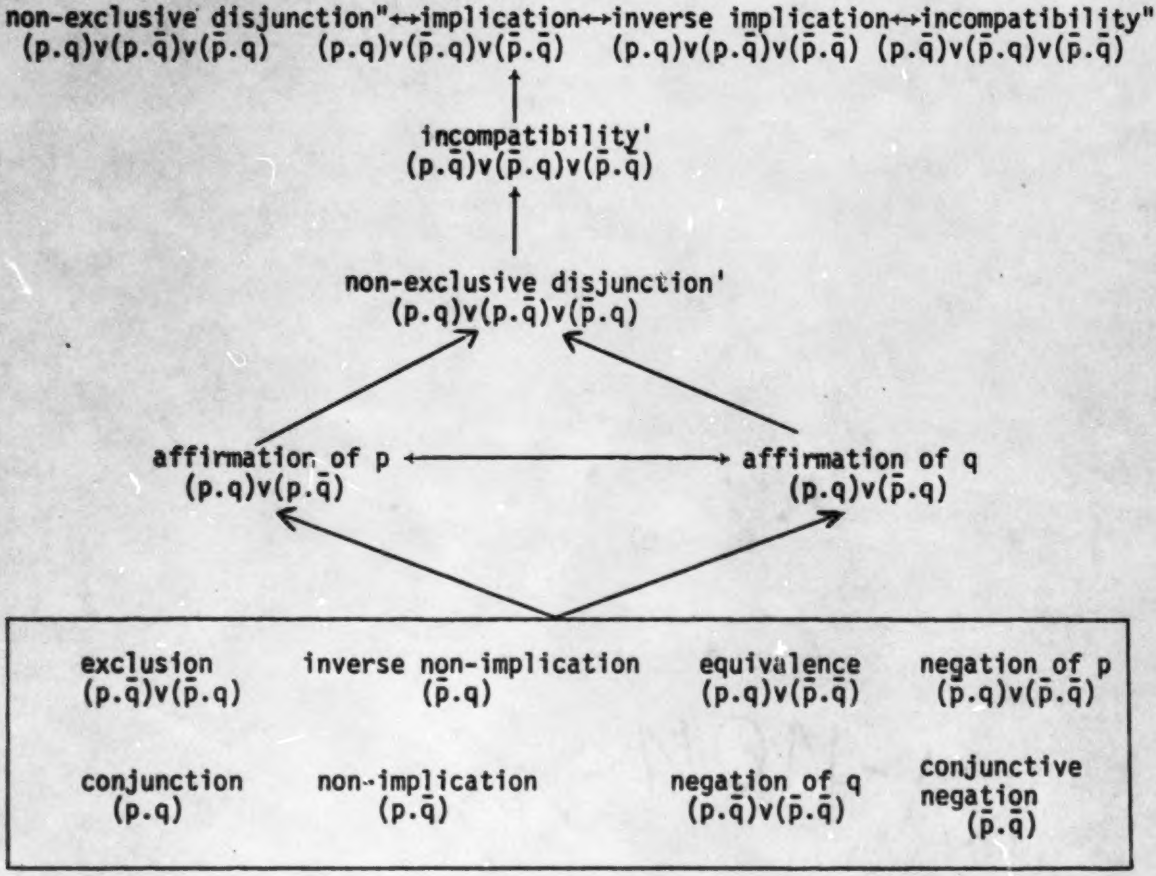

non-exclusive disjunction"↔implication↔inverse implication↔incompatibility"
$(p.q)v(p.\bar{q})v(\bar{p}.q)$ $(p.q)v(\bar{p}.q)v(\bar{p}.\bar{q})$ $(p.q)v(p.\bar{q})v(\bar{p}.\bar{q})$ $(p.\bar{q})v(\bar{p}.q)v(\bar{p}.\bar{q})$
equivalence
$(p.q)v(\bar{p}.\bar{q})$
incompatibility'
$(p.\bar{q})v(\bar{p}.q)v(\bar{p}.\bar{q})$
non-exclusive disjunction'
$(p.q)v(p.\bar{q})v(\bar{p}.q)$
affirmation of q
$(p.q)v(\bar{p}.q)$
+
negation of q
$(p.\bar{q})v(\bar{p}.\bar{q})$
affirmation of p
$(p.q)v(p.\bar{q})$
exclusion
$(p.\bar{q})v(\bar{p}.q)$
negation of p
$(\bar{p}.q)v(\bar{p}.\bar{q})$
inverse non-implication
$(\bar{p}.q)$
non-implication
$(p.\bar{q})$
conjunction
$(p.q)$
conjunctive negation
$(\bar{p}.\bar{q})$

Figure 3. Ordering among logical statements (Airasian, Bart & Greaney, 1975, p. 20) for 14 year old subjects.


non-exclusive disjunction"↔implication↔inverse implication↔incompatibility"
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$(p.q)v(\bar{p}.q)v(\bar{p}.\bar{q})$
$(p.q)v(p.\bar{q})v(\bar{p}.\bar{q})$
$(p.\bar{q})v(\bar{p}.q)v(\bar{p}.\bar{q})$
incompatibility'
$(p.\bar{q})v(\bar{p}.q)v(\bar{p}.\bar{q})$
non-exclusive disjunction'
$(p.q)v(p.\bar{q})v(\bar{p}.q)$
affirmation of p
$(p.q)v(p.\bar{q})$
affirmation of q
$(p.q)v(\bar{p}.q)$
exclusion
$(p.\bar{q})v(\bar{p}.q)$
inverse non-implication
$(\bar{p}.q)$
equivalence
$(p.q)v(\bar{p}.\bar{q})$
negation of p
$(\bar{p}.q)v(\bar{p}.\bar{q})$
conjunction
$(p.q)$
non-implication
$(p.\bar{q})$
negation of q
$(p.\bar{q})v(\bar{p}.\bar{q})$
conjunctive negation
$(\bar{p}.\bar{q})$

Figure 4. Ordering among logical statements, grade 9 subjects.
± = .10.

at least 40% of the subjects answered correctly up through the items dealing with non-exclusive disjunction and incompatibility.

The data of the replication study do not show the branching of the original nor is there the same differentiation among logical skills at the lower end of the hierarchy. There does exist the same trend and generally same placement of skills vertically. The differential placement of equivalence and the easier attainment of negation of q by subjects would appear to be due to the hints employed by the interviewers in this study and recommended by Ward (1972). These hints attend to items 1, 3, and 12 (conjunction, negation of q, and equivalence, respectively), thus causing a higher success rate on these items than would otherwise have been attained. Hence their lower placement within the hierarchy of this study if, in fact, such hints were not given in the earlier one -- and there is no indication in the reports that they were given.

An Extension of the Replication

Subjects, Instrumentation, and Procedures

This investigation broadened the age range and interviewed subjects at grades 7 and 11. The grade 7 subjects were chosen randomly from mathematics and science classes in the same Winnipeg junior high school as the grade 9 subjects. The grade 11 subjects were selected randomly from all grade 11 mathematics classes in a Winnipeg high school (grade 11 mathematics is a required subject). Fifty subjects were in each sample. Instrumentation and data analysis procedures were identical to those in the replication study.

Results

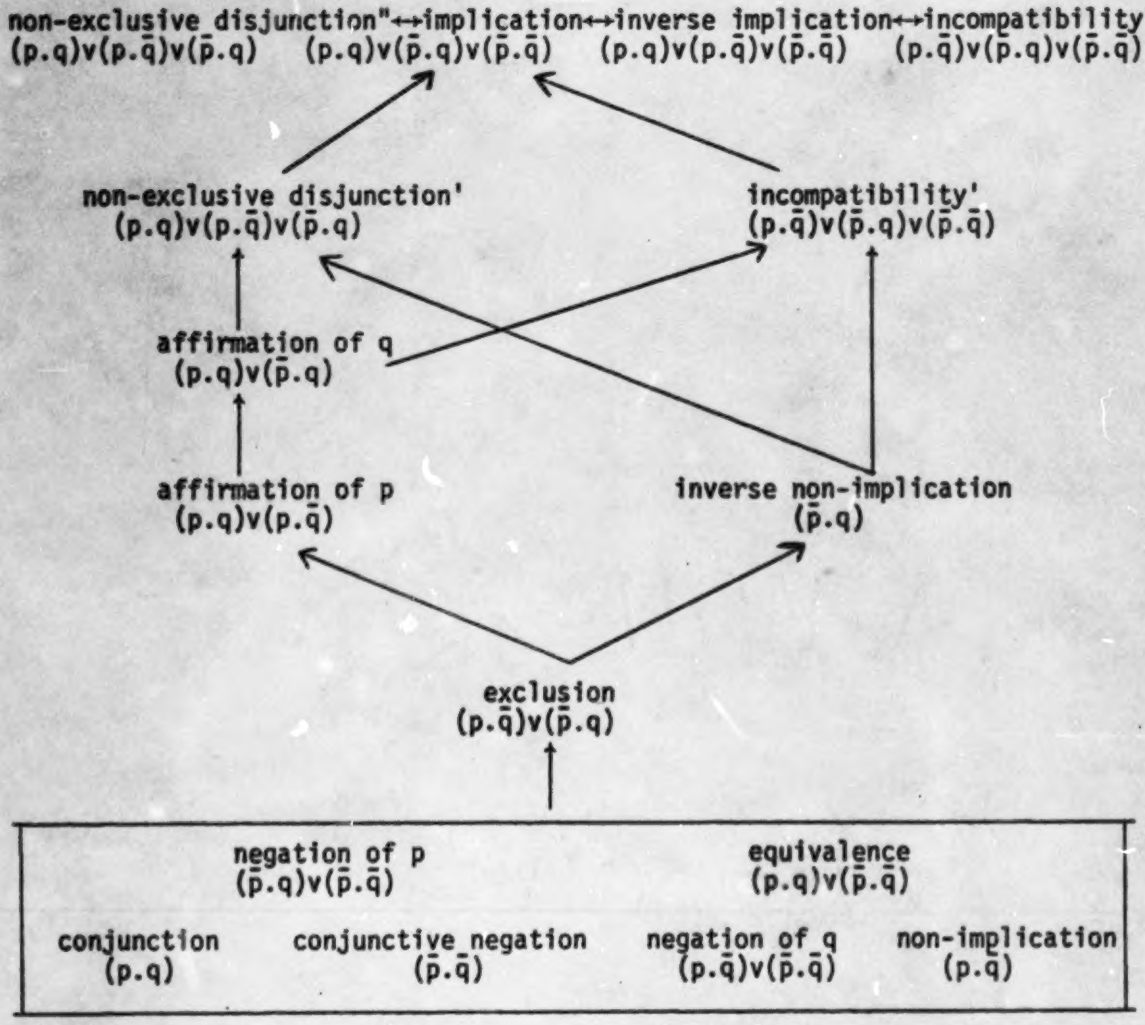
The logical orderings resulting from these data are shown in Figures 5 and 6. When compared to the grade 9 results, the seventh graders show the same general upward trends but with more complex branching and a few important differences. Both *exclusion* and *inverse non-implication* appear higher up in the tree, while *non-exclusive disjunction* and *incompatibility* (items 5 and 8 respectively) do not show the prerequisite relationship exhibited by grade 9 subjects. For seventh graders these two operations appear to be independent, with *inverse non-implication* and *affirmation of q* prerequisite to each, and each in turn is a prerequisite to top level logical operations. However, in three of the four sets of data at hand, *non-exclusive disjunction* does appear as a prerequisite to *incompatibility*.

Perhaps the results obtained from grade 11 (generally 16 years of age) subjects most nearly approximate those of the earlier study except that the lower elements of the hierarchy are equivalent. As before, the items of *equivalence* and *negation of q* are excepted from such a generalization because the results appear to be due to the hints.

In order to further investigate the measurement model, the data from all three grade levels were combined. While this might be a questionable procedure given the dissimilar trees which resulted from the by-grade-level analysis, it was thought that the larger N and a reduced tolerance level might give an interesting result and permit further exploration of the model. In this situation the tolerance level was set at $t = .05$ for $N = 150$; the resulting hierarchy is shown in Figure 7. It shows much more complex branching than any of the preceding hierarchies. It also shows several relationships to be quite different from that obtained with $t = .10$. In particular, it suggests that *affirmation of p*, *non-exclusive disjunction*, and *incompatibility* are independent, whereas at $t = .10$ these are prerequisite to each other in the order stated. Nevertheless, the same general upward trend is evident.

Discussion

Concommitant with the empirical studies described in the preceding two sections, it was an aim of this paper to examine the viability of ordering


non-exclusive disjunction"↔implication↔inverse implication↔incompatibility"
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$(p.q)v(p.\bar{q})v(\bar{p}.\bar{q})$
$(p.\bar{q})v(\bar{p}.q)v(\bar{p}.\bar{q})$
non-exclusive disjunction'
$(p.q)v(p.\bar{q})v(\bar{p}.q)$
incompatibility'
$(p.\bar{q})v(\bar{p}.q)v(\bar{p}.\bar{q})$
affirmation of q
$(p.q)v(\bar{p}.q)$
affirmation of p
$(p.q)v(p.\bar{q})$
inverse non-implication
$(\bar{p}.q)$
exclusion
$(p.\bar{q})v(\bar{p}.q)$
negation of p
$(\bar{p}.q)v(\bar{p}.\bar{q})$
equivalence
$(p.q)v(\bar{p}.\bar{q})$
conjunction
$(p.q)$
conjunctive negation
$(\bar{p}.\bar{q})$
negation of q
$(p.\bar{q})v(\bar{p}.\bar{q})$
non-implication
$(p.\bar{q})$

Figure 5. Ordering among logical statements, grade 7 subjects.
$\underline{+}$ = .10.

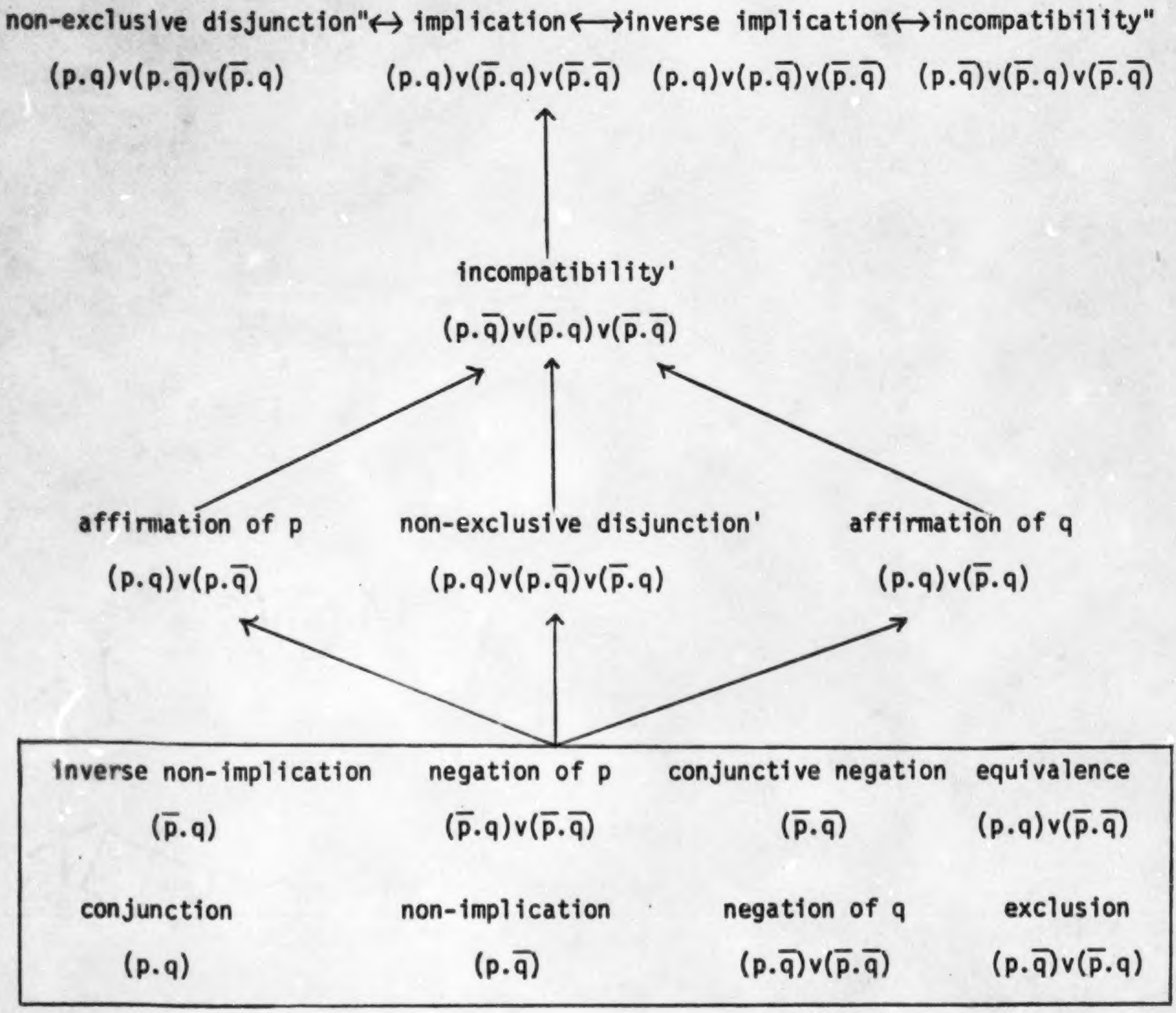
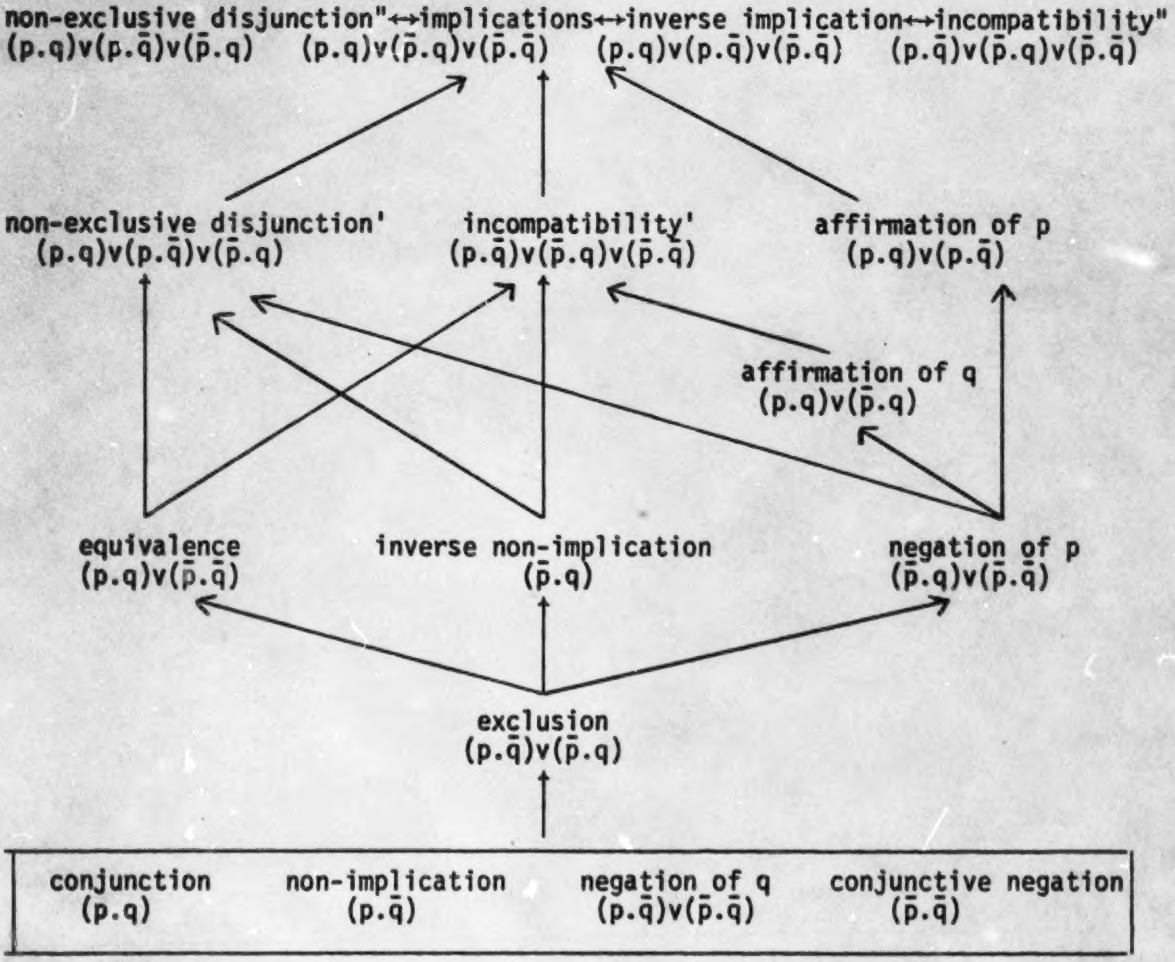

non-exclusive disjunction" ⟷ implication ⟷ inverse implication ⟷ incompatibility"
$(p.q)v(p.\overline{q})v(\overline{p}.q)$
$(p.q)v(\overline{p}.q)v(\overline{p}.\overline{q})$
$(p.q)v(p.\overline{q})v(\overline{p}.\overline{q})$
$(p.\overline{q})v(\overline{p}.q)v(\overline{p}.\overline{q})$
incompatibility'
$(p.\overline{q})v(\overline{p}.q)v(\overline{p}.\overline{q})$
affirmation of p
$(p.q)v(p.\overline{q})$
non-exclusive disjunction'
$(p.q)v(p.\overline{q})v(\overline{p}.q)$
affirmation of q
$(p.q)v(\overline{p}.q)$
inverse non-implication
$(\overline{p}.q)$
negation of p
$(\overline{p}.q)v(\overline{p}.\overline{q})$
conjunctive negation
$(\overline{p}.\overline{q})$
equivalence
$(p.q)v(\overline{p}.\overline{q})$
conjunction
$(p.q)$
non-implication
$(p.\overline{q})$
negation of q
$(p.\overline{q})v(\overline{p}.\overline{q})$
exclusion
$(p.\overline{q})v(\overline{p}.q)$

Figure 6. Ordering among logical statements, grade 11 subjects. ± = .10.


non-exclusive disjunction"↔implications↔inverse implication↔incompatibility"
$(p.q)v(p.\bar{q})v(\bar{p}.q)$
$(p.q)v(\bar{p}.q)v(\bar{p}.\bar{q})$
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non-exclusive disjunction'
$(p.q)v(p.\bar{q})v(\bar{p}.q)$
incompatibility'
$(p.\bar{q})v(\bar{p}.q)v(\bar{p}.\bar{q})$
affirmation of p
$(p.q)v(p.\bar{q})$
affirmation of q
$(p.q)v(\bar{p}.q)$
equivalence
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conjunction
$(p.q)$
non-implication
$(p.\bar{q})$
negation of q
$(p.\bar{q})v(\bar{p}.\bar{q})$
conjunctive negation
$(\bar{p}.\bar{q})$

Figure 7. Ordering among logical statements, grades 7, 9, and 11 combined. $\underline{t} = .05$.

theory as an approach to construction of learning hierarchies. This approach, as a measurement model, must provide an algorithmic procedure for determining the hierarchy. The set of relationship definitions appears to satisfy this requirement, with the establishment of a tolerance level allowing for error of measurement. Thus the algorithm is adequate.

If ordering theory is a viable approach, then the same results, i.e., the same ordering, would be expected to appear for any grade level or age group, except for larger or smaller sets of equivalent elements at the top or bottom ends of the hierarchy. In other words, if these logical skills are indeed logically ordered according to this prerequisite relationship, then the order should remain constant. This is a big _if_ and the reader should be cautioned here: The prerequisite relationship determined by this model is really a chronological one since no task analysis or other analytic procedure has established any other relationship between pairs of skills.

Data from three grade levels were collected and analyzed. A comparison of the three resulting hierarchies cannot be carried out statistically and one must rely on an analysis of the graphic relationships. Clearly the three trees are not identical, but nevertheless the general trends in the ordering do remain. Only further data will determine whether the results remain essentially constant. It would appear that samples larger than 50 would be appropriate. While the algorithm was able to handle this number within the definitions, many cases were decided on the basis of the equalities rather than the inequalities. This was one reason for examination of the total sample of 150 subjects. Thus, although the interpretation of that hierarchy in light of the individual results may be problematic, it provided a good test of the branching capabilities of the hierarchy construction procedure. It is also possible that with respect to the prerequisite relationship defined in this study, the learning order of the logical skills is not fixed. Researchers, including this one, are looking for regularities and patterns, but maybe they do not exist for these sixteen thinking operations as a set. They may exist only for subsets or perhaps not at all. Ennis (1975, 1976) would suggest that we are looking at the wrong operations.

Another variable not considered in this investigation is language. The fact that two elements of the set of sixteen appeared twice, in different linguistic forms, and placed differently, suggests such a variable. Would similar differential results have occured with another logic skill expressed in two forms? How might semantic and syntactic dimensions of such a linguistic variable be related? Such considerations also beg the question of the role of subject matter (in the "discipline" or "area of knowledge" sense). Damarin (1977) has begun a line of research which aims to examine the use of logical connectives by students specifically in mathematics.

Two further findings of the Airasian, Bart, and Greaney (1975) study do appear to be supported by the present results. First, disjunctive propositions such as implication and disjunction which are the unions of three atomic propositions are indicated to be more difficult than simpler unitary propositions such as conjunction and non-implication. Secondly, "understanding atomic propositions such as conjunction and non-implication is indicated to be prerequisite to understanding bi-atomic propositions such as equivalence and affirmation of p. Also, understanding bi-atomic propositions is indicated to be prerequisite to understanding tri-atomic propositions..." (p. 22). This finding does lend added support to the hypothesis that the psychological order of comprehension for propositons has an additive atomic basis.

In summary, the aims of the study have been met. A definition and a set of procedures for applying ordering theory have been presented. Both replication and extension studies have been reported. The question of the validity of the ordering theory approach to logical reasoning learning hierarchy construction has not been fully answered.

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